Mathematics Framework
Chapter 8: Mathematics: Investigating and Connecting, High School

Second Field Review Draft

The Crucial Mathematics of High School

The California Common Core State Standards for Mathematics (CA CCSSM) describe mathematics learning objectives for California high school students. During high school, students develop more maturity from which to exercise choice about their futures, and accordingly they have more opportunities to make choices that reflect their interests and aspirations. The CA CCSSM include learning goals for all students as well as “plus” standards for students whose interests and aspirations lead them during high school to a more intensive specialization in mathematics and related fields.

The CA CCSSM's "Higher Mathematics" (high school) content standards are organized in Conceptual Categories. These learning goals are described beginning on page 120 of the CA CCSSM.

- Number and Quantity
- Algebra
- Functions
- Modeling (the Modeling standards all appear *within* the other Conceptual Categories)
- Geometry
- Statistics and Probability

The Higher Mathematics Standards for Mathematical Practice (SMPs) are the same as for kindergarten through grade eight:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

As a carefully-constructed collection of learning goals, the CA CCSSM were never intended to be a design for *instruction*, and in fact the sheer number of standards (especially at the high school level) mean that a standard-by-standard approach to instruction is impossible.

The framework's role is to guide implementation of the CA CCSSM, not to simply restate or explicate its standards (learning goals). Thus, the framework is written from the perspective of instruction (both instructional materials and enacted instruction). This requires careful consideration of many issues in addition to learning goals: motivation, coherence, students' and teachers' cultural and linguistic assets, access and equity, context, sustainability, and many more.

In order to build from the CA CCSSM's learning goals (many of which are necessarily of small scale) to a description of mathematics to guide instruction—that is, a description that incorporates the many issues of instruction in addition to assessable mathematics content learning goals—this section integrates content and practice to illustrate the mathematical understandings, skills, and dispositions expected of all graduating students, with additional notes about students who aspire to pursue a college degree in STEM and quantitative fields, including computer science, data science, and finance.

For consistency across the entire transitional kindergarten through grade twelve span, the expected understandings, skills, and dispositions of graduates are organized by Content Connection (CC).

- Communicating Stories with Data (CC 1)
- Exploring Changing Quantities (CC 2)
- Taking Wholes Apart, Putting Parts Together (CC 3)
- Discovering Shape and Space (CC 4)

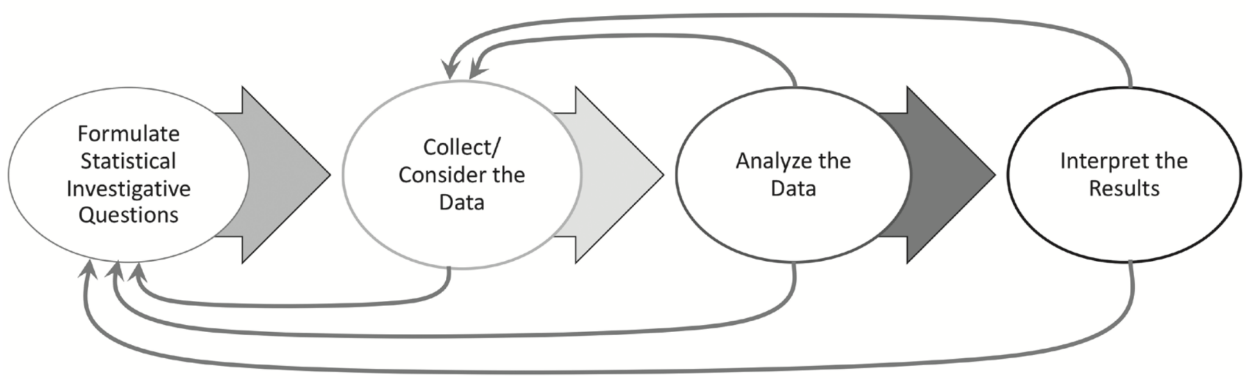
The important cross-cutting areas of *Modeling* and *Reasoning and Justification* cannot be understood as separate areas of content and practice; rather, the expected understandings, skills, and dispositions in these areas are discussed through all four Content Connections.

In the Mathematics: Investigating and Connecting pathway outline in subsequent sections, the Content Connections will again be used to organize the *types of investigations* in which students should engage—in order to build the understandings, skills, and dispositions described here.

Communicating Stories with Data (CC 1)

Most quantitative situations that graduates will encounter in their lives involve reasoning about and with data. This Content Connection is discussed in greater depth in Chapter 5 (Data Science). Broadly, high school graduates must understand the statistical problem-solving process and develop skills in each of its steps.

Figure 8.1. The Statistical problem-solving process (GAISE II)



By graduation, students should understand the important roles that questioning plays in every step of this process: Statistical investigative questions, data collection and interrogation questions, data analysis questions, and interpretation questions.

Formulate statistical investigative questions: Students must anticipate variability, and understand that random processes can produce data that varies in predictable ways in the aggregate (and thus understand that meaningful relationships between varying quantities might be discernible even from noisy data). Graduates should be able to formulate statistical investigative questions for the purposes of describing, comparing, and predicting, and propose ways to gather data to help answer those questions. Questions may involve several variables of interest and may concern questions of association (correlation) and causality.

Collect/Consider the Data: Graduates should propose ways to collect data to answer statistical investigative questions. They understand the difference between surveys, observational studies, and experiments; and can choose the option(s) best suited to the question of interest. They discuss possible sources of bias in surveys and in study design, and understand privacy and other ethical issues that accompany data collection and analysis. They understand the role that randomness plays in the ability to generalize (to a larger population) findings from surveys, observations, or experiments. For secondary data, graduates can ask questions about the origin of the data and its ability to help answer the statistical investigative question, including possible sources of bias.

Students whose interests and aspirations lead them to a more focused study of data science in high school will, in addition, know good practices for designing surveys, studies, and experiments—including issues of sample size and methods for random sampling and assignment. They will also understand practices for cleaning, organizing, and handling data.

Analyze the data: All graduates should be able to identify appropriate summaries (graphical displays, tables, summary statistics) for quantitative or categorical data, and to generate those summaries for some data sets using technology. For a relationship between two quantitative variables, they should be able to use appropriate technology to generate a correlation coefficient and a least-squares regression line, and then to interpret both in the context of the data. They understand that statistical claims about populations are based on probability.

This phase of the process and the previous (as well as CC 2) require that graduates understand the mathematics of measurement, including conversion between different units, the use of units that are rates (such as km/hr or people per square mile), and when it does or does not make sense to combine quantities (adding length and area makes no sense; dividing kilometers by hours might express something useful). In these measurement contexts, graduates use proportional reasoning and understand percentages and ratios as ways to express multiplicative comparisons and relationships between quantities.

Students specializing in data science will learn more advanced techniques for describing and representing relationships between variables, and considerably more of the probabilistic underpinning of statistical claims. This equips them to construct and interpret confidence intervals and *p*-values. They have developed the habit of using dimensional analysis to make sense of computations, and can manipulate ratios, percentages, and scientific notation in order to understand and express results.

Interpret results: Graduates can interpret the results of their analysis in the context of the statistical investigative question. They can explain the meaning of population estimates or other results, and discuss possible sources of error such as missing data and imperfect data collection. They are able to interpret margins of error and confidence intervals, demonstrating correct probabilistic understanding. They can communicate their results via writing, speaking, and visual representations.

Students specializing in data science can interpret *p*-values, demonstrating an understanding of the probabilistic claim that an observed result is not plausible under a particular set of assumptions. They use technology to decide the most important predictor variables for a variable of interest in a multivariable situation. This summary of expected learning for students specializing in quantitative areas is consistent with “Level C” expectations in the Pre-K–12 Guidelines for Assessment and Instruction in Statistics Education II (GAISE II) from the American Statistical Association and National Council of Teachers of Mathematics. Along with an understanding of statistical methods, those who aim to enter a data science major in college should also have experience with programming.

Exploring Changing Quantities (CC 2)

Reading and writing with mathematics involves recognizing quantities in situations; translating relationships between them from natural language, visual, or other forms into mathematical forms (often equations, but also graphs, tables, and more); working with and moving between these mathematical forms to understand or answer questions about the relationships; and interpreting findings back in the original context. All students should develop this inclination and ability to a significant degree. Most standards in the *Functions* conceptual category are included here; some regarding building functions are discussed in CC 2. Most *Modeling* work involves this process of identifying and relating quantities in a situation.

Noticing and naming quantities in situations is key for students to understand that mathematics arises in—and helps to understand, explain, and solve problems in—situations that they wonder about (SMP.1). Students should all develop this ability to recognize and name quantities throughout their transitional kindergarten through grade twelve experiences, so that the high school task is to maintain and expand, rather than rediscover and redevelop, this inclination and ability. Graduates should be able to notice and name quantities in situations ranging across science, social science, mathematics, everyday life, and more.

Describing relationships between quantities in mathematical forms, and being able to flexibly work with and move between those forms, is central to using mathematics to reason about situations and questions of interest (SMP.4). To describe a relationship, especially in order to predict one quantity from one (or more) other quantities, often requires that a *function* of one (or, eventually, more than one) quantity be expressed. Understanding the concept of a function and interpreting functions in context is a major outcome of high school mathematics.

During high school, all students should learn to recognize and represent linear, exponential, and logarithmic relationships in multiple forms (graphs of functions, algebraic formulas, scatter plots, tables, recursive rules, and verbal descriptions), to use appropriate technology, and to move flexibly between these representations as necessary to understand, explain, or solve problems in the situation. Students should also be able to use and recognize quadratic functions as models for important physical phenomena, such as motion under the force of gravity, and to describe properties of quadratic functions that differ from those linear and exponential functions.

Students should also be able to recognize periodic phenomena and to adjust the period, amplitude, horizontal shift, and vertical shift of a trigonometric function (perhaps experimentally, via a computer algebra system) to represent simple periodic relationships. More discussion of modifying functions in this way is in *Taking Wholes Apart, Putting Parts Together* below. Graduates should also understand trigonometric functions as ways to describe the ratios between different side lengths in right triangles, and that these ratios are invariant under similarity.

Much of the power of mathematics as a lens for understanding authentic contexts and problems lies in the fact that the same mathematics (when abstracted from the particular quantities in the current context) applies to such varied situations. Thus, when students understand exponential functions, they can use them to reason about population growth, interest-bearing monetary accounts, and radioactive decay, to name just a few.

All high school graduates should be able to apply reasoning about linear, quadratic, and exponential functions across a variety of contexts, and interpret that abstract reasoning in the particular quantities of those contexts (SMP.2). Students should understand abstraction as a way to reason similarly across different contexts (SMP.8). For example, the contexts of population growth, interest-earning accounts, and radioactive decay were not designed to be applications of exponential functions; rather, exponential functions are noticed, described, defined, and studied because of the observed similarity in reasoning about these (and many more) contexts.

Students whose interests and aspirations lead them to a more focused study of mathematics during high school are expected to develop both a larger vocabulary of familiar function types, and more depth and flexibility in using them to model phenomena and solve problems (often using technology). In particular, they can use and manipulate trigonometric functions to represent and explore periodic phenomena, and rational functions to represent ratios between two varying quantities (rates). Most college-level study in mathematics will expect considerable familiarity and comfort with manipulating algebraic expressions and equations and modeling with functions in order to solve problems and make certain features of functions apparent.

Taking Wholes Apart, Putting Parts Together (CC 3)

The Conceptual Categories *Algebra* and *Number and Quantity* largely fall into this Content Connection, along with portions of the *Functions* and *Geometry* Conceptual Categories that involve relating a mathematical object to its constituent parts or building a new object from others.

Across many contexts and typically-separated areas of mathematical content, students must develop the inclination and ability to see the component parts of complex situations, functions, geometric objects, etc.; to investigate those components; and to assemble observations about the components into understanding about the original setting. CC 3 can also be seen as assembling and communicating the steps in a solution, in justifying a claim or answer in a learning group, or in forming hypotheses from observations. In these ways, students develop their ability to reason logically. Logical reasoning is at the heart of mathematical discovery, communication, and connection, and students' initial understanding of the role of proofs, as ways to explain the validity of facts, is predicated upon their ability to reason visually, symbolically, concretely and abstractly.

The Conceptual Category *Algebra* (as distinct from *Functions*, in CC 2 above) describes graduates' expected abilities to see structure in expressions (considering the contributions of, and interpreting, different parts such as terms and factors), create equations to describe relationships (often by separately representing different contributions to varying quantities, and combining those contributions into one equation), and reason with equations (and inequalities) in order to understand situations and solve problems. Manipulating expressions and equations are tools for reasoning with equations and inequalities. Familiarity with arithmetic properties, used in decomposing and composing numerical quantities in earlier grades, provides the foundation upon which students can understand the purpose and import of algebraic properties, not as arbitrary laws to be memorized, but as distillations of ideas already familiar to them.

The high school *Number and Quantity* standards include extending properties of exponents from natural number exponents to rational exponents, and extending the concept of number to include complex numbers. Graduates should understand that properties encoding *observations* in one system (such as $(a^b)^c = a^{(bc)}$, for a real number a and whole numbers b and c) can be used to *define* the meaning of similar symbols in other systems (such as $5^{(1/3)}$, with a non-whole number exponent). Similarly, extending the real numbers to the complex numbers is accomplished by extending desired properties from the real numbers to a larger set (one in which $x^2 = -1$ has a solution).

In both *Geometry* and *Functions*, graduates understand the many ways that functions are built up from simpler ones or from defining properties—for example, rigid transformations from translations, rotations, and reflections (add dilations for similarity transformations); linear (resp. exponential) functions from a starting value (y-intercept) and a constant additive (resp. multiplicative) rate of change. Modifying functions via horizontal and vertical shifts, vertical and horizontal reflections, and vertical and horizontal compression/stretching are further examples; graduates should be able to identify the effects of the various algebraic replacements, and choose appropriate one(s) (e.g., in graphing software) to produce functions with desired characteristics (e.g., to model data).

In *Geometry*, understanding the whole from its parts plays more roles: Informal arguments for the area and volume of various objects by dissection arguments; relationships between three-dimensional objects and one- or two-dimensional figures (cross-sections, faces, edges).

Students who specialize in mathematics may also understand that vectors and matrices are additional objects that can name new types of quantities, and can be manipulated to understand those quantities, using operations similar (but not identical) to those of real numbers.

Discovering Shape and Space (CC 4)

This Content Connection contains the bulk of the *Geometry* Conceptual Category, as well as some trigonometric functions standards in *Functions*.

Graduates should understand congruence and similarity of plane figures in terms of transformations of the plane, and understand that measurement-based criteria for congruence—such as angle-side-angle for triangles—follow from the transformation definitions. They should understand why all length measures scale by the same factor under a similarity transformation. They understand that these definitions of congruence can be used to prove many facts about lines, angles, and shapes; and they connect tools of formal constructions with rigid motions to establish the validity of constructions.

Ratios of corresponding sides of triangles should be understood to be preserved by similarity transformations. For right triangles, then, these trigonometric ratios are properties of the *angles* in the triangle (since one of the acute angles defines a right triangle up to similarity). Graduates should be able to identify similar right triangles in applied settings, and use trigonometric ratios and the Pythagorean Theorem to find unknown measurements in right triangles in terms of known sides and angles. They know that the domains of the functions sin(x), cos(x), and tan(x) can be extended to all real numbers using the unit circle, giving periodic functions that can be used to model phenomena (see CC 2 above).

Students should understand that all circles are similar, and know that relationships between various angle measures and length measures in a circle can be used to find others.

The coordinate plane must be understood as a tool for connecting geometry and algebra, by providing equations that describe geometric objects, as well as geometric objects that describe (the solutions to) equations in two variables. Graduates know that some geometric facts are most easily established using algebraic representations, and that geometric observations can lead to better understanding in the algebraic context.

Students whose interests and aspirations lead to more focused mathematics work in high school may also extend their tools for analyzing triangles to non-right triangles by deriving the Laws of Sines and Cosines, and a formula for the area of a general triangle in terms of side and angle measures; and using these to find unknown measurements in triangles.

Standards for Mathematical Practice

In addition to the areas of content to be covered, the practice of mathematics is described in the CA CCSSM through the Standards for Mathematics Practice (SMPs, http://www.corestandards.org/Math/Practice). Designing instructional time so that students are engaging and building proficiency in these practices is crucial. Each SMP is described with a paragraph in the CA CCSSM; here only the titles are listed.

SMP.1. Make sense of problems and persevere in solving them.

SMP.2. Reason abstractly and quantitatively.

SMP.3. Construct viable arguments and critique the reasoning of others.

SMP.4. Model with mathematics.

SMP.5. Use appropriate tools strategically.

SMP.6. Attend to precision.

SMP.7. Look for and make use of structure.

SMP.8. Look for and express regularity in repeated reasoning.

The Importance of a Renewed Focus on High School Mathematics

California students' demonstration of deep mathematical learning on local and state assessments continues to be a concern and a priority for districts. This includes the importance of high levels of mathematics understanding for college and career preparedness. Both the National Assessment of Educational Progress (NAEP) and the Programme for International Student Assessment (PISA) provide compelling data supporting a renewed focus on high school mathematics education. The National Council of Teachers of Mathematics (NCTM) summarized these findings this way:

> The steady improvement in mathematics learning seen since 1990 at the elementary and middle school levels has not been shared at the high school level, underscoring the critical need for change in mathematics education at the high school level.
>
> *Catalyzing Change in High School Mathematics* (NCTM, 2018)

Since 2000, US math performance has steadily declined in both absolute and relative terms on the international PISA exams sponsored by the Organization for Economic Cooperation and Development and now ranks 32rd in the world, far below the average. (See chapter 1.) In contrast to the highest-achieving countries, US performance is lower for both high- and low-achievers and shows much wider gaps, which are more closely related to socioeconomic status. As a consequence, calls for reform in mathematics education have been widespread.

Mathematics in the highest-achieving countries is typically taught in heterogenous classrooms prior to tenth grade, and, in high school, in an integrated fashion with domains of mathematical study combined to allow for more robust conceptualization and problem solving, rather than in a sequence in which Algebra, Geometry, Algebra II / Trigonometry are taken separately, one by one. For example, in Japan, the highest-scoring country on the most recent PISA exams, math I, II, and III each combine elements of algebra, geometry, measurement, statistics, and trigonometry. The focus is on taking time for students to intently discuss and collaboratively solve complex problems that are represented in multiple ways—often just one complex problem in a class period—rather than memorizing formulas and applying rote procedures to a large set of problems that students often do not deeply understand (Okano and Tsuchiya, 1999, Stigler and Hiebert, 1997). Reforms over the last decade have focused more intently on experiential and project-based learning and applications to real-world problems by adding data uses to each grade level (Ministry of Education, 2010). When differentiation occurs at tenth grade to add greater challenge to the courses of advanced students, the curriculum remains similar, and both lanes allow students to reach advanced courses like calculus.

A similarly integrated curriculum is used in Korea, the second ranked country on PISA, where a “learner-centered” approach advanced by the Ministry has focused mathematics on active engagement in problem solving. There, too, students take the same integrated set of courses through grade ten (each of which integrates content from 6 domains: 'Numbers and Operations', 'Geometric Figures', 'Measuring', 'Probability and Statistics', 'Letters and Expressions', and 'Patterns and Functions,' with basic and enriched content within each course to meet students’ interests and needs). They choose “electives” in eleventh and twelfth grade, such as additional integrated courses or statistics, calculus, discrete mathematics, or practical mathematics (Paik, 2004).

In Estonia, the third ranked and most rapidly improving country, the curriculum integrates arithmetic and measurement along with geometric, algebraic, and statistical concepts throughout the grades and has a strong focus on modeling and solving word problems in all domains, including with algebraic tools (see National Center on Education and the Economy, n.d.; and Hemmi, Brating, and Lepik, 2020). A set of reforms over the last decade has focused intensely on the use of computers and descriptive statistics for data analysis throughout the grades, and the use of real-world problems to organize mathematical inquiry (Holm, Hommik, and Kikas, 2016).

In Finland, also one of the highest performing countries on PISA, students work in heterogenous classes on a common curriculum during the first nine years of their education, using the approach set out in this framework that teaches mathematics as a set of big ideas and connections in ways that value student ideas and curiosity (Sahlberg, 2021). Finnish students outperform US students by a considerable margin. In eighth grade 15.3 percent of Finish students score at the highest levels in Program for International Student Assessment (PISA) mathematics tests compared to only 8.8 percent of students in the United States (PISA, 2012).

As noted in chapter 1, these curriculum approaches are consonant with what researchers are learning from neuroscience about how the brain works as it develops mathematical understanding using multiple representations and approaches, productive inquiries, and connections to real-world problems that are engaging and allow a more integrated approach to problem solving. These approaches also inform this framework, described below.

Designing Instruction for Equitable and Engaging High School Mathematics

Five Components of Equitable and Engaging Teaching

This framework's Chapter 2 (Teaching for Equity and Engagement) is structured around five components of equitable and engaging teaching, which are briefly revisited here. The components should inform high school instructional design as much as earlier grades. For much fuller discussions, refer to Chapter 2.

1. Plan Teaching Around Big Ideas: Mathematics is a subject made up of important ideas and connections. Curriculum standards tend to divide the subject into smaller topics, but it is important for teachers and students to think about the big ideas that characterize mathematics at their grade level and the connections between them. The big ideas for high school are set out later in this chapter.
2. Use Open, Engaging Tasks: When questions are narrow and focused, only some students are cognitively challenged at an appropriate level, and the questions are often not very interesting. When tasks are open, they allow all students to work at levels that are appropriately challenging for them, within the content in their grade.
3. Teach Toward Social Justice: Teachers can take a justice-oriented perspective while broadening access to and interest in math at any grade level, kindergarten through grade twelve, by choosing examples that connect math to questions that are relevant and important to students, helping them feel belonging (Brady et al.,

2020), and empowering them with tools to address important issues in their lives and communities.

4. Invite Student Questions and Conjectures: One of the most important yet neglected mathematical acts in classrooms is that of students asking or posing mathematical questions. These are not questions to help students move through a problem; they are questions that are sparked by wonder and intrigue (Duckworth, 2006).
5. Center Reasoning and Justification: Reasoning is fostered when students have the opportunity to talk about mathematics with each other through whole class discussions and small group work on open tasks.

These components of instruction remain important at the high school level, and for many high school educators they will represent a change from their own high school experience.

Planning Instruction to Drive Investigation and Make Connections

Since motivating students to care about mathematics is crucial to forming meaningful content connections, this Framework identifies three **Drivers of Investigation** (DIs), which provide the “why” of learning mathematics, to pair with the Standards of Mathematical Practice (SMPs—the “how” of learning and doing mathematics) and four **Content Connections** (CCs), which provide the “what” of mathematics (the high school CA CCSSM content standards) to be learned in an activity. So, the Drivers of Investigation propel the learning of the content framed in the Content Connections.

Content Connections

The four CCs described in the framework, described in detail above, provide mathematical coherence through the grades:

- Content Connection 1: Communicating Stories with Data
- Content Connection 2: Exploring Changing Quantities

- Content Connection 3: Taking Wholes Apart, Putting Parts Together
- Content Connection 4: Discovering Shape and Space

Drivers of Investigation

The CCs should be developed through investigation of questions in authentic contexts; these investigations will naturally fall into one or more of the following DIs. The DIs are meant to serve a purpose similar to that of the Crosscutting Concepts in the California Next Generation Science Standards (CA-NGSS), as unifying reasons that both elicit curiosity and provide the motivation for deeply engaging with authentic mathematics. In practical use, teachers can use these to frame questions or activities at the outset for the class period, the week, or longer; or refer to these in the middle of an investigation (perhaps in response to the “Why are we doing this again?” questions that often crop up), or circle back to these at the conclusion of an activity to help students see “why it all matters.” Their purpose is to pique and leverage students’ innate wonder about the world, the future of the world, and their role in that future, in order to foster a deeper understanding of the Content Connections and grow into a perspective that mathematics itself is a lively, flexible endeavor by which students can appreciate and understand so much of the inner workings of our world. The DIs are:

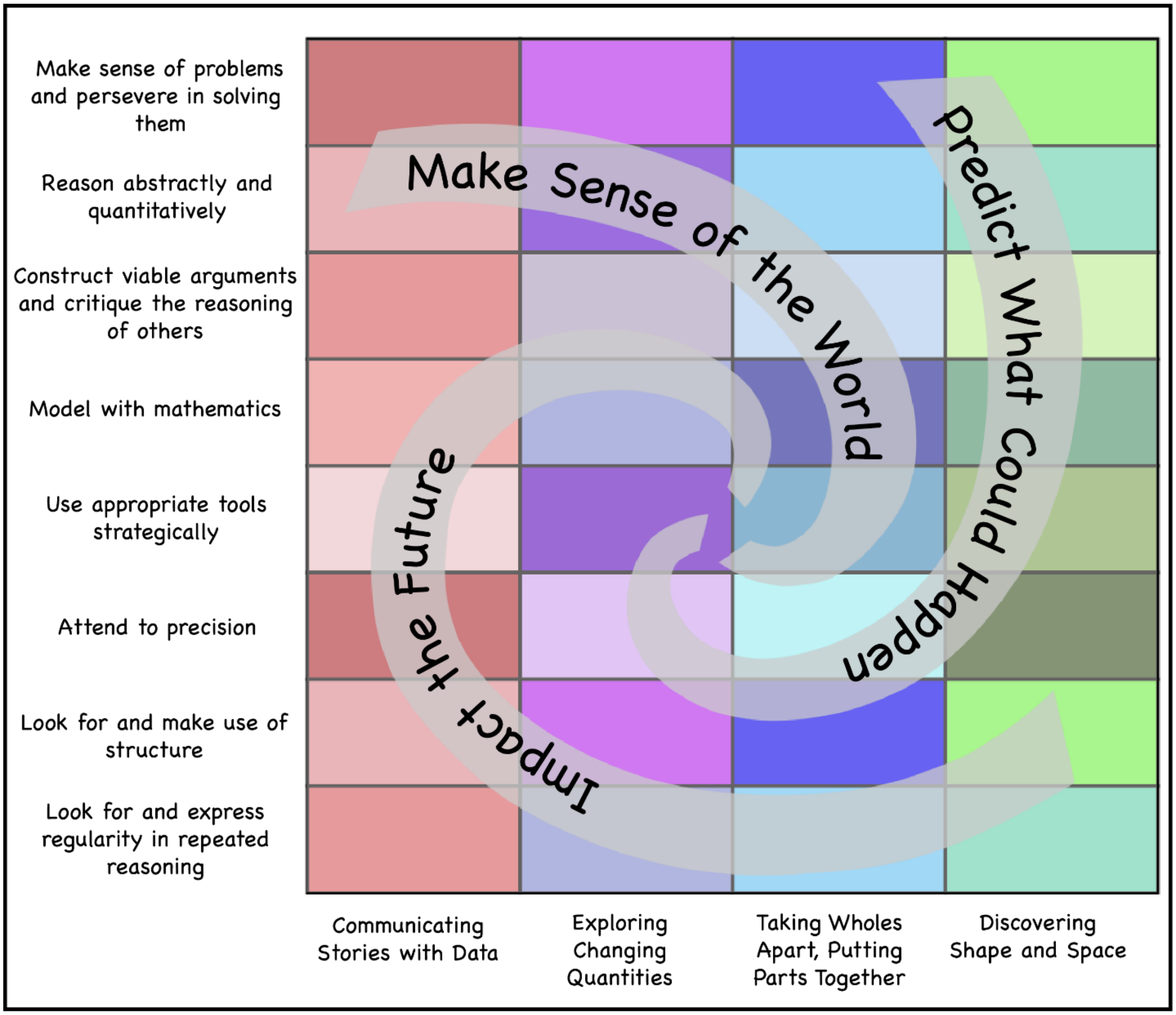
- Driver of Investigation 1: Make Sense of the World (Understand and Explain)
- Driver of Investigation 2: Predict What Could Happen (Predict)
- Driver of Investigation 3: Impact the Future (Affect)

Lesson ideas that drive design of instructional activities will link one or more SMPs with one or more Content Connections in the context of a Driver of Investigation, so that students can (for example) Model with mathematics *while* Communicating Stories with Data *in order to* Predict What Could Happen. Or students can Reason Abstractly and Quantitatively *while* Exploring Changing Quantities *in order to* Impact the Future. The aim of the Drivers of Investigation is to ensure that there is always a reason to care about mathematical work —and that investigations allow students to make sense, predict, and/or affect the world. The table below is a simple way to begin planning instructional activities:

Standards for Mathematical Practice The "How"	**Content Connections The "what"**	**Drivers of Investigation The "Why"**
Students will… SMP.1. Make Sense of Problems and Persevere in Solving them SMP.2. Reason Abstractly and Quantitatively SMP.3. Construct Viable Arguments and Critique the Reasoning of Others SMP.4. Model with Mathematics SMP.5. Use Appropriate Tools Strategically SMP.6. Attend to Precision SMP.7. Look for and Make Use of Structure SMP.8. Look for and Express Regularity in Repeated Reasoning	**while…** CC1. Communicating Stories with Data CC2. Exploring Changing Quantities CC3. Taking Wholes Apart, Putting Parts Together CC4. Discovering Shape and Space	**in order to…** DI1. Make Sense of the World (Understand and Explain) DI2. Predict What Could Happen (Predict) DI3. Impact the Future (Affect)

The following diagram is another illustration of the ways that the Drivers of Investigation relate to Content Connections and Mathematical Practices, as cross-cutting themes. Any Driver of Investigation can be matched with any Content Connection(s) and Mathematical Practices; the diagram should not be interpreted to imply that each possible SMP-CC-DI combination should have activities designed around it.

Figure 8.1: Content Connections, Mathematical Practices and Drivers of Investigation



Link to long description

Instructional materials should primarily involve tasks that invite students to make sense of these big ideas, elicit wondering in authentic contexts, and necessitate mathematical investigation. Big ideas in math are central to the learning of mathematics, link numerous mathematical understandings into a coherent whole, and provide focal points for students' investigations. An authentic activity or problem is one in which students investigate or struggle with situations or questions about which they actually wonder. Lesson design should be built to elicit that wondering. For example, environmental observations and issues on campus and in students' local community provide rich contexts for student investigations and mathematical analysis. Such discussions will concurrently help students develop their understanding of California’s Environmental Principles and Concepts.

The framing of “Students will (SMP) while (CC) in order to (DI)” helps teachers and curriculum writers to focus instruction on the big ideas, shown in Appendix A. It is similar to the way that the CA NGSS’s seven Crosscutting Concepts serve as themes which span multiple grades and are present in the various sciences.

Within each Content Connection, students’ experiences should first emerge out of exploration or problems that incorporate student problem-posing (Cai and Hwang, 2019). Meaningful student engagement in identifying problems of interest helps increase engagement even in subsequent teacher-identified problems. Identifying contexts and problems before solution methods are known makes explorations more authentically problematic for students, as opposed to simply exercises to practice previously learned exercise-solving paths.

A well-known example of the difference between a stereotypical use of problems and the one assumed in this pathway is described in Dan Meyer’s TED Talk (Meyer, 2010): Meyer considers a standard textbook problem about a cylindrical tank filling from a hose at a constant rate. The textbook provides several sub-steps (area of the base, volume of the tank), and the final question “How long will it take to fill the tank?” The task appears at the end of a chapter in which all the mathematical tools to solve the problem are covered; thus, students experience the task as an exercise, not an authentic problem.

In the problem-based technique advocated here, the tank-filling context is presented prior to any introduction of methods or a general class of problems, in some way that authentically raises the question, “How long will it take to fill?” and preferably in a way that has a meaningful answer available for a check (e.g., a video of the entire tank-filling process, as in the TED Talk). After the question has been raised (hopefully by students), students make some estimates, and then the development of the necessary mathematics is seen as having a purpose. Viewing the end of the video prompts meta-thinking about process (*Why is our answer different than the video shows?*) much more effectively than a “check your work” prompt or a comparison with the answer in the back of the book. This tank-filling problem could occur in the “Exploring Changing Quantities” Content Connection of MIC 1 (see the Mathematics: Investigating and Connecting pathway in Appendix A), Integrated I, or Algebra 1. Note that the problem integrates linear function and geometry standards.

As this example shows, the problem-embedded learning envisioned in this framework does not imply a curriculum in which all learning takes place in the context of large, multi-week projects, though that is one approach that some curricula pursue. Problems and activities that emphasize a big idea-based approach as outlined here can also be incorporated into instruction in short time increments, such as 45-minute lessons or even in shorter routines such as Think-Pair-Share, or Math Talks (see Chapter 3). There are a number of lesson plan formats which take a problem-embedded approach, including one from Los Angeles Unified School District which adopts a three-phase lesson structure incorporating student question-posing, solving, and reflecting stages (LAUSD, n.d.).

Because mathematical ideas and tools are not neatly partitioned into categories, many clusters of standards appear in multiple Content Connections. For example, the Quantities cluster *Reason quantitatively and use units to solve problems* (Q.A) is a set of standards that will be built and reinforced in many investigations based in data and varying quantities; hence this cluster is included in both Content Connection 1 (Communicating stories with data) and Content Connection 2 (Exploring changing quantities).

A more extensive investigation that cuts across several Content Connections is illustrated in this climate change vignette.

Vignette: Exploring Climate Change

Course: MIC 1/Integrated Math I

Content Connection 2: Exploring Changing Quantities

Driver of Investigation 3: Impact the Future (Affect)

Domains of Emphasis: HS.S.IC, HS.S-ID

SMPs: SMP.1, 2, 3, 4

Background Reading on Climate Change

With the beginning of the Industrial Revolution of the in the mid-1700s, the world began to see many changes in the production of goods, the work people did on a daily basis, the overall economy and, from an environmental perspective, the balance of the carbon cycle. The location and distribution of carbon began to shift as a result of the Industrial Revolution, and have continued to change over the last 250 years as a result of the growing consumption of fossil fuels, industrialization, and several other societal shifts. During this time, the distribution of carbon among Earth's principal reservoirs (atmosphere; the oceans; terrestrial plants; and rocks, soils, and sediments) has changed substantially. Carbon that was once located in the rock, soil, and sediment "reservoir," for example, was extracted and used as fossil fuels in the forms of coal and oil to run machinery, heat homes, and power automobiles, buses, trains, and tractors. (This provides a good opportunity for discussing and reinforcing California Environmental Principle IV. "The exchange of matter between natural systems and human societies affects the long-term functioning of both.") Before the Industrial Revolution, the input and output of carbon among the carbon reservoirs was more or less balanced, although it certainly changed incrementally over time. As a result of this balance, during the 10,000 years prior to industrialization, atmospheric CO_2 concentrations stayed between 260 and 280 parts per million (ppm). Over the past 250 years human population growth and societal changes have resulted in increased use of fossil fuels, dramatic increase in energy generation and consumption, cement production, deforestation and other land-use changes. As a result, the global average amount of carbon dioxide hit a new record high of 407.4 ppm in 2018—with the annual rate of increase over the past 60 years approximately 100 times faster than previously recorded natural increases.

The "greenhouse effect" impacts of rising atmospheric CO_2 concentrations are diverse and global in distribution and scale. In addition to melting glaciers and ice sheets that many people are becoming aware of, the impacts will include sea level rise, diminishing availability of fresh water, increased number and frequency of extreme weather events, changes to ecosystems, changes to the chemistry of oceans, reductions in agricultural production, and both direct and indirect effects on human health. (This offers a good opportunity to reinforce California Environmental Principle II. "The long-term functioning and health of terrestrial, freshwater, coastal and marine ecosystems are influenced by their relationships with human societies.")

Mathematics/Science/English Languages Arts/Literacy (ELA) Task:

Determine the relative contributions of each of the major greenhouse gases and which is the greatest contributor to the global greenhouse effect and, therefore, should be given the highest priority for policy changes and governmental action. Examine the growth patterns of related human activities and their relative contributions to release of the most influential greenhouse gas. Based on these factors, analyze the key components of the growth patterns and propose a plan that would reduce the human-source release of that greenhouse gas by at least 25–50 percent, and determine how that change would influence the rate of global temperature change.

Classroom Narrative:

Mathematics, science, and language arts teachers met to co-plan this interdisciplinary task. They each felt that the task was challenging and authentic, requiring students to draw from different disciplines to forge a solution, just as is done in the real world. They developed a sequence of activities to get the students started, being careful not to over-scaffold the task or to give students too much guidance toward possible solutions pathways, but ensuring their work supplemented and supported the larger task.

Launch: Student teams are provided with the task and then read the article “Climate Change in the Golden State” (https://californiaeei.org/media/1329/greenhouse-cc.pdf) to gather evidence about the scale and scope of the effects of climate changes in California. As this is an extended text, the ELA teacher offers guidance on how to access this document using a screen reader. This support aligns with the Universal

Design for Learning (UDL) principle—Provide multiple means of representation. The ELA teacher also provides an interactive note-taking guide for students to use. Students highlight parts that are not clear, they note important claims made by the authors, and formulate their own questions to share in groups. Students ask: Who is most affected if we do not try to fix problems related to climate change? Who is most affected if we do? Should we care about climate change? Students use their reading and research skills as basis for tackling the question of climate change.

Orienting Discussion: The class discusses four key questions:

1. Why do temperatures seem to be increasing? What are possible causes?
2. Can the recent changes in California's climate be explained by natural causes?
3. If natural causes cannot explain the rising temperatures, what other factors have produced these changes?
4. If temperatures in California's climate continue to rise, what effects will this have on humans and the state's natural systems?

Having read and processed the key article, students start to unpack these questions. Students look up the meaning of "anthropogenic," then rephrase the questions in their own words to see if they understand the meaning. Both the reading and the initial class discussion prepare students to push forward.

Motivated to help reduce climate change in California and globally, students decide to break down their task into more manageable pieces:

1. Determining the major greenhouse gases;
2. Analyzing the relative contributions of each gas and deciding which is the greatest contributor to global climate change and thus should be given the highest priority for policy changes and governmental action;

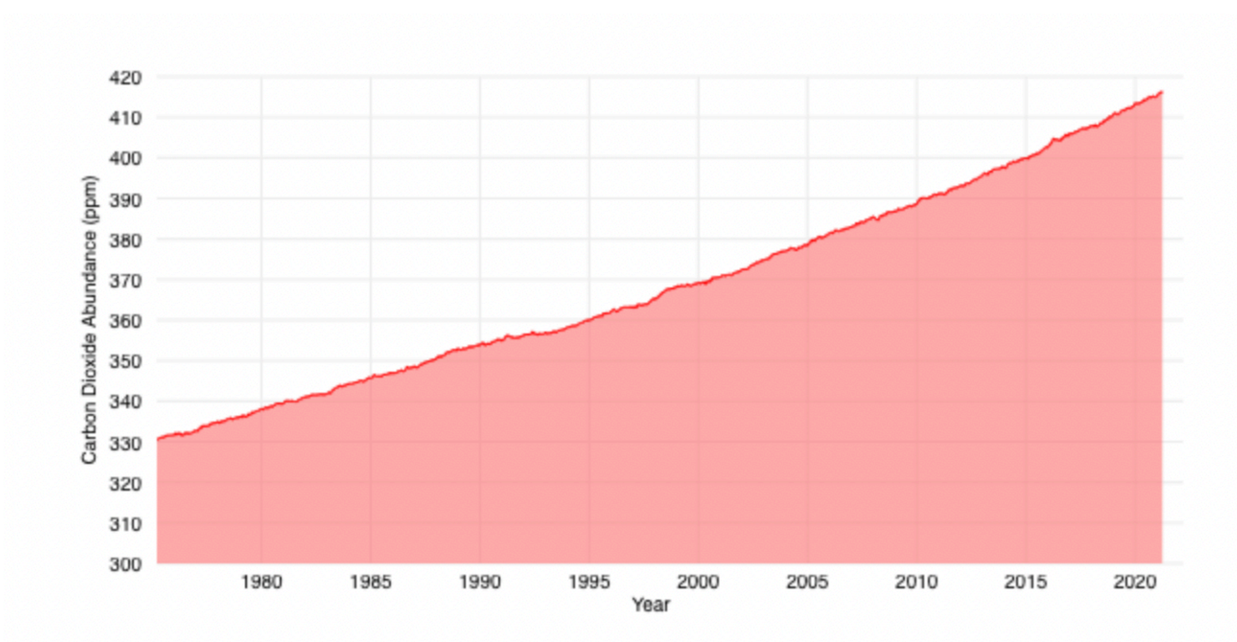
3. Collecting data on the human activities that cause increases to the release of the most influential greenhouse gas;
4. Analyzing the key components of the growth patterns of this gas;
5. Based on influences to the growth pattern, developing a plan to reduce the human-source release of that greenhouse gas by 25–50 percent; and,
6. Determining how their plan would influence the rate of global climate change.

Team Research

Students start researching online, using familiar criteria to vet the trustworthiness of the data sources.

They visit https://www.climate.gov and the California Air Resources Board (https://ww2.arb.ca.gov) to gather most of the data they need.

At https://www.climate.gov they discover a graph that shows the influence of the major human-produced greenhouse gases from 1980–2020.



https://www.climate.gov/news-features/understanding-climate/climate-change-atmospheric-carbon-dioxide

Looking at the graph and prompted by the teacher's questions, "What do you notice? What do you wonder?" students wonder about various aspects and implications. They jot these wonderings down and then speak in small groups. They notice that all major contributing gases seem to be increasing over time, though some say CFC-11 isn't obviously increasing; and others note that CFC-12 seems to have leveled out around 1990. Some students question this, as both still look like they are "going up" on the graph; this disagreement and ensuing discussion helps all students make sense of the graph.

Through a process of collaboration, they work together to synthesize their questions into coherent and meaningful inquiries:

1. Why are there labels on both vertical axes? What do the three labeled axes represent?
2. Why is there a labeled 43-percent increase? An increase in what? Over what time frame? How was this calculated?
3. What does this data display suggest is the most important greenhouse gas?
4. How does the year-to-year growth change over these 38 years?

Most teams choose to focus their efforts on reducing CO2 emissions based on the graph above. One team decides to work with methane because they believe that CO2 emissions are harder to reduce, and they believe they can make a bigger difference by reducing methane emissions. The increased autonomy accessed this unit empowers students to explore and allow the results of those explorations to direct them—not typical instruction in math, science, or ELA. The teachers work with some groups that may struggle with the openness of the task. Teachers encourage students to build from and explore each other's ideas.

Each team researches the sources of human emissions of the gas they have chosen, uses their understanding of political and psychological opportunities and barriers to decide on most-likely policy shifts to achieve the desired 25–50 percent reduction in emissions, and prepares a presentation for the class outlining their solutions. The teaching team provides additional expertise to help interpret the complexity of the information students are collecting and synthesizing.

Team Presentations

As teams prepare for their presentations, they return to the driving question of the task. From all the data they collected, they must now distill the most important information to describe their analysis and recommendations. Part of each presentation is a version of the National Oceanic and Atmospheric Association graph above, extended into the future with the assumed implementation of the team's proposal. Calculating the impact of their proposal on the rate of temperature change will require interpreting the left vertical axis label on the graph. The teaching team videotapes the presentations and reports to capture the range of practices that students are using such as quality of their research, analysis of data, effectiveness of their visuals, and clarity of their report, given audience, and purpose.

After all teams have presented, the final activity is to put all the pieces together to address the following big idea: What will be the impact on climate change if all the teams' proposals are implemented?

The Need for Integration in High School Mathematics

Children are naturally curious about their world and the environment in which they live, and this curiosity fuels their desire to wonder, describe, understand, and ask questions. Similar to how a child responds to these curiosities, learning mathematics develops through attempts to describe, to understand, and to answer questions. Mathematics provides a set of lenses for viewing, describing, understanding, and analyzing phenomena; and for solving problems—such as local issues related to environmental and social justice, through engineering design practices (CA NGSS HS-ETS1-2)—which might occur in the “real world” or in abstract settings such as within mathematics itself. For instance, finance, the environment, and science all offer phenomena, such as recurrent patterns or atypical cases, which are better understood through mathematical tools; such phenomena also arise *within* mathematics (see Chapter 4, for instance).

However, mathematics is never developed in order to answer questions about which the explorer is *not* curious; and *learning* mathematics is not much different. By experiencing the ways in which mathematics can answer natural questions about their world, both in school and outside of it, a student’s perspectives on both mathematics and their world are integrated into a connected whole.

Definition of Integration

There are multiple contexts for which the term “integrated” has been used in connection with mathematics education. In this chapter, “integrated” refers both to the connecting of mathematics with students’ lives and their perspectives on the world, and to the connecting of mathematical concepts to each other. This reference to both can result in a more coherent understanding of mathematics. Integrated tasks, activities, projects, and problems are those which invite students to engage in both of these aspects of integration. All three of the pathways described in Appendix A can incorporate both aspects of integration: opportunities that are relevant to students and their experiences, and opportunities to connect different mathematical ideas.

Studies have found that the integration of mathematical topics through authentic problems that draw from different areas of mathematics can increase engagement and achievement (Grouws, Tarr, Chávez, Sears, Soria, and Taylan, 2013; Tarr, Grouws, Chávez, and Soria, 2013).

Motivation for Integration

Critique the effectiveness of your lesson, not by what answers students give, but by what questions they ask.

—Fawn Nguyen (2016), Mesa Union School District, junior-high mathematics teacher

In keeping with the thrust of this framework, all high school curriculum and instruction can benefit from thoughtful approaches which leverage relevance to students with opportunities to reveal fundamental connections among related topics. A guiding question for measuring these two aspects in classroom activities, in any course, is “Can I see evidence that students wonder about questions that will help to motivate learning of mathematics and that connect this learning to other knowledge?”

Designing Instruction with Integration in Mind

The primary challenge for the design of any high-school pathway is to bridge the gap between the CA CCSSM’s lists of critical content goals and the difficult tasks teachers face every day when providing instruction that casts mathematics as a subject of connected, meaningful ideas, that can empower students to understand and affect their world.

As described in Chapter 2, it is important that exploration and question-posing occur *prior to* teachers telling students about questions to explore, methods to use, or solution paths. A compelling experimental research study compared students who learned calculus actively, when they were given problems to explore before being shown methods, to students who received lectures followed by solving the same problems as the active learners (Deslauriers, McCarty, Miller, Callaghan, and Kestin, 2019). The students who explored the problems first learned significantly more (see also Schwartz and Bransford, 1998). However, despite the increased understanding of the exploratory learners, students in both groups *believed* that the lecture approach was more effective—as the active learning condition caused them to experience more challenge and uncertainty. The study not only showed the effectiveness of students exploring problems before being taught methods, but the value of sharing with students the importance of struggle and of thinking about mathematics problems deeply.

In a similar vein, different conceptions and unfinished learning add value to classroom discussions when they can be made visible and used thoughtfully. Activities should be designed to elicit common mis- or alternative conceptions, not to avoid them. This requires that teachers work through tasks before using them in classes, in order to anticipate common responses and plan ways to value contributions and use them to build all students' understanding. The goal of mathematics class must be deeper understanding and more flexibility in using and connecting ideas—*not* quicker answer-getting (Daro, 2013).

Other research examines beliefs and attitudes such as utility value (belief that mathematics is relevant to personal goals and to societal problems), and this research shows a severe drop off in utility value during high school (Chouinard and Roy, 2008). However, teaching methods that increase connections between course content and students' lives, and that include careful focus on effective groupwork, can significantly increase utility value for students (Cabana, Shreve, and Woodbury, 2014; Boaler, 2016a, 2016b, 2019; Hulleman, Kosovich, Barron, and Daniel, 2017; LaMar, Leshin, and Boaler, 2020).

Pathways in Grades Nine Through Twelve

Pathways of mathematics courses in grades nine through twelve provide opportunities for students to develop a disposition toward reasoning and communication in mathematics, knowledge of mathematical ideas and skills, and the ability to think both critically and creatively in solving problems. In any of the pathways supported in California, the approach of integration amongst topics, described in detail in the prior section, is highly valued, as are the other recurrent themes of this framework: focusing on big ideas and active investigation. Illustrations of these types of investigations are provided in the last section of this chapter.

The Starting Point for High School Coursework

As the framework outlines 3 potential pathways for California students, it begins with the foundation of the CA Common Core 6, 7 and 8 courses, which were set out in the initial framework as the best middle school preparation, with grade eight offering algebra content integrated with challenging content in other areas of mathematics that strengthen and deepen students' foundation for more advanced mathematics. Evidence suggests that this content supports success in Common Core mathematics, and can prepare most students to successfully take courses through Advanced Statistics or Calculus if they so choose.

Some students will be ready to accelerate into Algebra I or Integrated Mathematics I in eighth grade, and, where they are ready to do so successfully, this can support greater access to a broader range of advanced courses for them. At the same time, successful acceleration requires a strong mathematical foundation. Research indicates that in the era in which California policy encouraged all students to take Algebra in eighth grade, success for many students was undermined. Given that this experiment was designed in part to enable students to reach Calculus by the end of high school, it could be preferable to adjust the high school curriculum, eliminating redundancies in the content of current courses, or organize supplemental course taking in summer programs, to allow students who wish to take Calculus after completing Algebra or Integrated Mathematics I in ninth grade to be able to do so successfully.

Currently, most high schools require courses in Algebra, Geometry, Algebra 2, and Pre-calculus before taking a course in Calculus, or a pathway of Integrated courses 1, 2, 3, then Pre-calculus. This sequence means that students cannot easily reach Calculus unless they have taken a high school algebra course in middle school. This has led to many students missing the structured content of middle school mathematics, often by skipping the grade eight course, or by taking compressed courses. Among the problems with this approach is that some students who take eighth grade Algebra instead of the CA Common Core grade eight course may miss foundational learning, and those who do not take that course are filtered out of the calculus pathway early on, with significant racial and gender inequalities (Joseph, Hailu, and Boston, 2017). Moreover, English learners have disproportionately less access, are placed more often in remedial classes and are steered away from STEAM courses and pathways (National Academies of Sciences, Engineering, and Medicine, 2018).

Since achieving a solid foundation in mathematics is more important for long-term success than rushing through courses with a superficial understanding, it would be desirable to consider how students who do not accelerate in eighth grade can reach higher level courses, potentially including Calculus, by twelfth grade. One possibility could involve reducing the repetition of content in high school, so that students do not need four courses before Calculus. Algebra 2 repeats a significant amount of the content of Algebra 1 and Pre-calculus repeats content from Algebra 2. While recognizing that some repetition of content has value, further analysis should be conducted to evaluate how high school course pathways may be redesigned to create a more streamlined three-year pathway to pre-calculus / calculus or statistics or data science, allowing students to take three years of middle school foundations and still reach advanced mathematics courses.

While it should continue to be possible for students who are interested and ready to take Algebra I in eighth grade to do so, the experiment of rushing to Algebra in middle school without an opportunity to build readiness left a considerable trail of failure for many students, suggesting that should not be the only pathway by which students can reach higher level mathematics. In 2008, to incentivize districts to require eighth grade Algebra, the state's Board of Education voted to make the Algebra California Standards Test (CST) the "sole test of record" for the state's eighth graders. This vote required eighth graders to demonstrate proficiency on the state's end-of-course Algebra standards exam to satisfy accountability expectations under the No Child Left Behind Act and California's Public Schools Accountability Act (Rosin et al., 2009). Although this mandate was never fully implemented due to court challenges, many districts did make dramatic changes in course-taking in response to the change in the accountability system, which remained for several years.

Several studies found that, contrary to the hoped-for improvements, widespread acceleration led to significant declines in overall mathematics achievement. A study by Liang, Heckman, and Abedi (2012) found that approximately 60% of students who took Algebra in the eighth grade failed to score "proficient" on the end-of-course Algebra CST. Furthermore, students who failed eighth-grade Algebra and thus took the Algebra CST again at the end of their ninth-grade year scored lower on average than students who took the Algebra CST for the first time at the end of ninth grade.

A case study of a large California district that dramatically increased eighth grade Algebra enrollment rates found declines in student mathematics achievement (Domina et al., 2014). And a cross-district study of all California K–12 public school districts found that those that enrolled more students in eighth grade Algebra had large negative effects on student achievement on the math portion of the high school exit (CAHSEE) exam that students took in tenth grade (Domina et al., 2015).

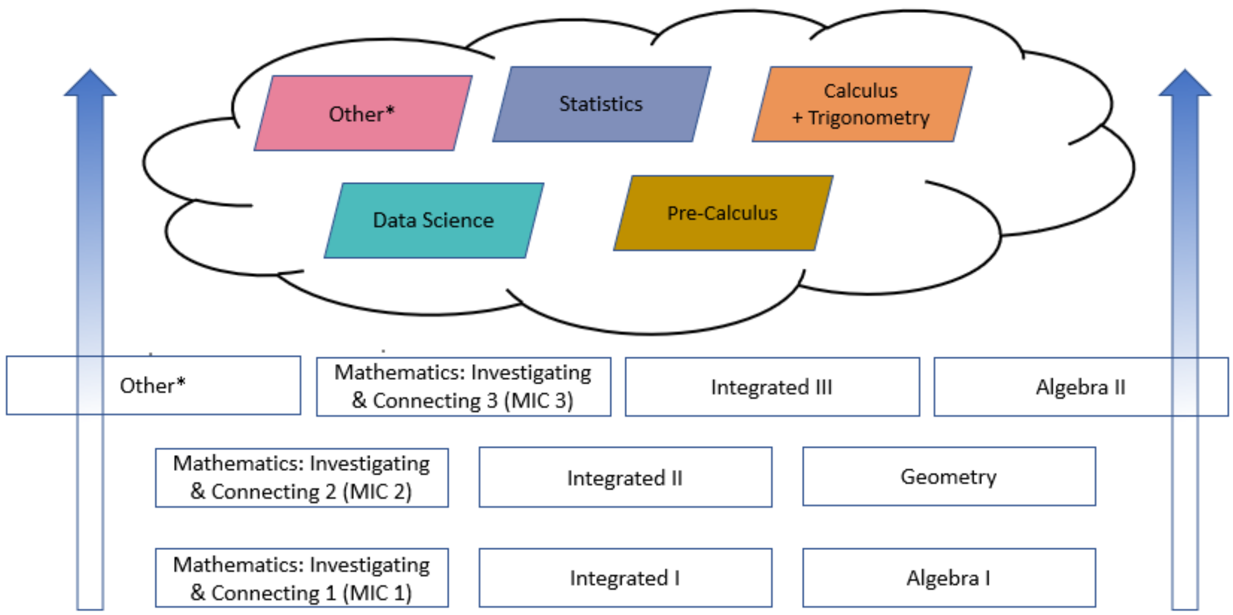
These challenges are no doubt a function of curricular readiness—having had the right foundations—and the quality of teaching both before and during the course itself. For schools that offer an eighth grade Algebra course or an Integrated Math I course as an option in lieu of Common Core Math 8, both careful plans for instruction that links to students' prior course taking and an assessment of readiness should be considered. Such an assessment might be coupled with supplementary or summer courses that provide the kind of support for readiness that Bob Moses' Algebra project has provided for underrepresented students tackling Algebra in middle and high schools for many years (Moses and Cobb, 2002).

Pathways in High School

High schools are free to organize their mathematics pathways in different ways. Figure 8.3 below indicates three possible pathways for high-school coursework, reflecting a common ninth- and tenth-grade experience, and a broader array of options in eleventh and twelfth grade. High schools will typically offer one of the first-two-years pathways (Integrated, MIC, or Traditional), and an array of more advanced courses. Choices made by students after their first two years should not lock them into any particular path: third-year courses should prepare students for all fourth-year courses to enable students' access to higher level mathematics as their interests and efforts develop. Whichever pathway is selected by a school, advanced students may complete that pathway in an accelerated fashion to access additional advanced mathematics courses, or, as described in chapter 9, they may be offered additional or supplemental challenges within or beyond the courses they take in their pathway. In addition to descriptions of the pathways courses, the appendix offers a discussion of the concepts that should be included for students intending to major in a STEM field of study in college.

Descriptions of the three Pathways, and the Big Ideas within each, are provided in Appendix A.

Figure 8.3



[Link to long description]()

In the diagram, other* Indicates the many types of other courses such as financial algebra, data science, or statistics with algebra, offered by high schools. Depending on district policy, most of these courses will require prerequisite knowledge of Integrated I and 2, MIC 1 and 2, or Algebra I and Geometry. See the following section.

Third- and Fourth-Year Courses

In addition to offering Integrated III, Algebra II or MIC 3, districts have the flexibility to offer other third-year courses. One example that is already offered by some districts (and is University of California A–G approved) is Financial Algebra, in which students engage in mathematical modeling in the context of personal finance (this course is comparable in rigor to an Integrated 3 or Algebra II course; it is not the same as a

“Consumer Math” or “Accounting and Finance” class currently offered by some schools, which are not UC A–G approved). Through this modeling lens, they develop understanding of mathematical topics from advanced algebra, statistics, probability, precalculus, and calculus. Instead of simply incorporating a finance-focused word problem into each Algebra 2 lesson, this course incorporates the mathematics concept when it applies to the financial concept being discussed. For example, the concept of exponential functions is explored through the comparison of simple and compound interest; continuous compounding leads to a discussion of limits; and tax brackets shed light on the practicality of piecewise functions. In this way, the course ignites students' curiosity and ultimately their engagement. The scope of the course covers financial topics such as: taxes, budgeting, buying a car/house, (investing for) retirement, and credit, and develops algebra and modeling content wherever it is needed. “Never has mathematics seemed so relevant to students as it does in this course,” says one teacher.

Another third-year course currently offered by several districts is a Data Science course. Data Science and Statistics need some discussion: Statistics is the science of collecting, displaying, analyzing, and drawing conclusions from data. Data Science is a newer field which uses tools of statistics, computer programming, and machine learning to extract meaning and understanding from (typically very large) data sets. There is much overlap between the terms. In the K–12 landscape, statistics courses often focus on statistical tools that allow data analysts to make claims about likelihood, correlation, estimates, confidence intervals, and the like. Data science courses usually have a broader focus on reasoning with data, including issues such as formulating investigative questions; gathering, interrogating, and cleaning data; and producing and interpreting visualizations of data, in addition to standard statistical tests and estimates.

Because data science is less well-defined in the K–12 landscape, some data science courses are constructed to develop (some) Integrated Math III content within the course, while others might require students to already have encountered the full Integrated Math I–III content. This is why Data Science appears as both a third year and a fourth-year course in Figure 8.3. However, note that the MIC 3 course described below is **not** a data science course, even if it is implemented using data-driven investigations, as the student learning outcomes of MIC III are given by the Integrated III content outline of the CA CCSSM.

Any of these third-year courses could lead to a range of fourth-year options as set out in the course diagram above (Figure 8.3). If students take another third-year course (besides MIC 3, Integrated Math III, or Algebra 2), they should be made aware that they are leaving the traditional pathway for taking Calculus in high school or in their first semester of college (as is sometimes expected for many STEM majors). While many colleges and universities accept a wide range of mathematical backgrounds, and provide pathways for students in STEM majors to complete Calculus in their first year, others expect to see incoming STEM majors having completed the content of MIC 3/Integrated III/Algebra 2 followed by a precalculus and/or calculus course.

College Expectations and Sample Student Pathways

By completing Algebra 1 and Geometry, Integrated I and II, or Mathematics: Investigating and Connecting (MIC) 1 and 2, students will satisfy the requirements of California Assembly Bill 220 of the 2015 legislative session that requires students to complete two mathematics courses in order to receive a diploma of graduation from high school, with at least one course meeting the rigor of Algebra 1. Depending upon their post-secondary goals, students may choose different third- and fourth-year courses, and all college-intending students should complete four years of mathematics in high school to meet California State University and University of California recommendations. Giving students a choice of pathways through their last two years of high school can elevate a student’s real-world application of mathematics understanding.

The variety of pathways reflect the many different interests and aims of students, such as those seeking employment directly after high school, others whose objective is a career in STEM for whom a university degree is critical, others who are interested in a university degree in a non-STEM intensive major, and the many students who are still deciding upon post-high school ambitions while they are in high school. The following scenarios illustrate a small sample of the different pathways students may take:

- Josef is planning to work in a fabrication shop after graduation, so he chooses to follow MIC 1 and 2 with a course in modeling and CAD to gain an understanding of the mathematics of die-casting and three-dimensional printing.
- Roscoe’s family has a business in which they plan to work after high school. In talking with a counselor, they realize that an accounting degree would enable Roscoe to oversee the business finances in the future. After Algebra 1, Geometry, and Algebra 2, Roscoe takes a Financial Algebra course, which enables them to get a solid start on understanding the underlying principles in the introductory finance courses at the collegiate level.
- Yesenia is planning to study political science, so she chooses a Data Science course in the third year (one which has Integrated I and II, MIC 1 and 2, or Algebra I and Geometry as prerequisites) and an AP Statistics course in her fourth year. This preparation serves her well, as she better understands the mathematics behind polling, apportionment, and gerrymandering from her Data Science course, as well as being well-equipped to understand the research methods in her political science courses from the Statistics course. In addition, since the Statistics course has an AP designation, she is well on her way to completing the General Education quantitative reasoning requirement for her university coursework.
- Ash is interested in working construction after high school but is also aware that his local community college offers a two-year certificate in construction management. Although he doesn’t pass Algebra I as a freshman, fortunately, his high school offers a support course, and with the extra time and attention, Ash passes Algebra I as a sophomore. His counselor advises him to take Geometry as a junior, since the study of shapes, angles, and measurement is beneficial for his career. Also, he could then take Algebra II as a senior, which provides the background to take trigonometry at the community college, a required course for the certificate.
- Inez likes digital photography, so was planning on majoring in graphic design at a university, a degree not requiring calculus. As Inez is completing her third-year course in Data Science, however, she found herself enjoying using the software

and various applications to work with the data sets and create captivating data displays. This, combined with her interest in creating mods for her favorite video game, has her now thinking about pursuing computer science coursework at a university. So, in her fourth year, she enrolls in her school's precalculus class, along with a half-semester support class her school offers for students whose interest in mathematics grows late in their high school time. She enters her university well-prepared to take freshman calculus and the programming classes she hopes to pursue alongside additional work in data science.

- Kai is interested in robotics engineering and was able to take Integrated I and II in junior high, and Integrated III during the first year of high school. By completing Precalculus in the second year, Kai is able to take AP Calculus in the third year. This enables multiple options to be available for Kai's fourth year, such as taking her school's data science course, or a programming and data science course at the local community college, multivariable calculus or other college courses.

Like Inez, students who decide to switch pathways (at high schools that offer multiple paths), can take advantage of the increasing flexibility afforded to those planning to enter a university upon graduation, in terms of which courses count for admission. In October 2020, the University of California (UC) system updated the mathematics (area C) course criteria and guidelines for the 2021–22 school year and beyond (University of California, 2020). The update includes the allowance of courses in Data Science to serve as the required third (or recommended fourth) year of mathematics coursework. For additional information on Data Science, see Chapter 5.

Overall, the revisions are to

- Clarify UC system expectations for college-prep mathematics courses that will help students acquire specific skills to master the subject's content and also gain proficiency in quantitative thinking and analysis;
- Support the efforts of high schools to develop and implement multiple college-prep mathematics course options for students; and

- Encourage the submission of a broader range of advanced/honors math courses (e.g., Statistics, Introduction to Data Science) for area C approval.

Key highlights of the policy updates:

- Courses that substantially align with Common Core (+) standards (see chapters on *Higher Mathematics Courses: Advanced Mathematics* and *Higher Mathematics Standards by Conceptual Category* in Standards for Mathematical Practice (SMPs) in the California Common Core State Standards: Mathematics (2013), and are intended for eleventh- and/or twelfth-grade levels are eligible for area C approval and may satisfy the required third year or recommended fourth year of the mathematics subject requirement if approved as an advanced mathematics course.
  Examples of such courses include, but are not limited to, applied mathematics, computer science, data science, pre-calculus, probability, statistics, and trigonometry.
- Courses eligible for UC honors designation must integrate, deepen, and support further development of core mathematical competencies. Such courses will address primarily the (+) standards of Common Core-aligned advanced mathematics (e.g., statistics, pre-calculus, calculus, or discrete mathematics).

The entire revised UC mathematics (area C) course criteria are located at https://hs-articulation.ucop.edu/guide/a-g-subject-requirements/c-mathematics/.

The California State University (CSU) system has developed several courses for the fourth year of high school (and some for earlier grades) which meet the area C (Mathematics) requirement for admission to the CSU. The CSU Bridge Courses page (http://cmrci.csu-eppsp.org/) lists mathematics/quantitative courses and projects working within the CSU system focused on supporting mathematics and quantitative reasoning readiness among K–12, CSU, and community-college educators. The courses emphasize subjects such as modeling, inference, voting, informatics, financial decision making, introduction to basic calculus concepts, connections among topics, theory of games, cryptography, combinatorics, graph theory, and connecting statistics with algebra. These courses have been adopted throughout the state in coordination with district and school initiatives to increase the variety of rich high-school mathematics coursework at the upper-grade levels.

As this framework has recommended, it would ultimately be desirable for high schools to be able to organize their course offerings to enable more students to get a strong foundation in middle school, without accelerating before many are ready, and reach higher level courses while in high school should they so desire. At the same time, mathematicians in colleges have begun to recognize the trade-offs that can occur when students rush through mathematics without a deep understanding. The fact that the majority of students who take calculus in high school repeat the course or take a lower-level course in college has led mathematicians such as Bressoud (2017) to state that the high school curriculum “does not appear to be meeting the needs of the students who have been accelerated” (2017, 5).

Indeed, in a large national study across 133 institutions, Sadler and Sonnert (2018) found that mastery of the mathematics considered preparatory for calculus had, on average, more than double the positive impact of taking a high school calculus course on students' later performance in college calculus.

The Mathematical Association of America (MAA) and NCTM issued a statement to urge that “the ultimate goal of the K–12 mathematics curriculum should not be to get into and through a course of calculus by twelfth grade, but to have established the mathematical foundation that will enable students to pursue whatever course of study interests them when they get to college” (Bressoud, 2012). The UC Board of Admissions and Relations with Schools (BOARS) made a similar statement:

> BOARS also strongly urges students not to race to calculus at the cost of full mastery of the earlier math curriculum. BOARS commends the Common Core's goal of deeper understanding of the mathematical concepts taught at each K–12 grade level. A strong grasp of these ideas is crucial for college coursework in many fields, and students should be sure to take enough time to master the material. Choosing an individually appropriate course of study is far more

> important than rushing into advanced classes without first solidifying conceptual knowledge. Indeed, students whose math classes are at a mismatched level—either too advanced or too basic—often become frustrated and lose interest in the topic. (BOARS, 2016).

This statement and UC's 2020 policy shift (Johnson, 2020) encouraging more flexibility in high school courses show the commitment of the University of California to value a range of mathematics courses as pathways to college. For some students—particularly those intending to major in mathematics, engineering and other STEM fields, a pathway to calculus is valuable. Many other students with different future intentions, such as social science degrees, may be better served with courses that lead to data science and statistics. Such courses should be designed so that they can also lead to a possible future in STEM. They are inherently mathematical and can be designed to include the topics enumerated at the beginning of this chapter and the competencies described as desired for entering college students by the University of California, California State University, and Community College system (Intersegmental Committee of the Academic Senates of the California Community Colleges, the California State University, and the University of California, 2010, 2013):

1) Modeling
2) Problem Solving
3) Developing analytic ability and logic
4) Experiencing mathematics in depth
5) Appreciating the beauty and fascination of mathematics
6) Building confidence
7) Communicating
8) Becoming fluent in mathematics

These competencies are reflected in the approach of this framework. Modeling is central to data science (see Chapter 5), and all of the competencies are developed through the mathematics approach described in other chapters. Colleges and universities point out that in developing “fluency” the goal is understanding, through which fluency can develop, a message that is also underlined in this framework. As described in the section below, deep understanding and fluency are best acquired when students can approach mathematics in an integrated manner that allows them to make connections across mathematical domains and with their lives, while accessing a range of tools to solve problems.

Four Vignettes

Each of the four vignettes in this section illustrates teaching approaches which can be utilized in a variety of courses and within any of the three pathways presented in Appendix A. Each vignette demonstrates a Content Connection. For a more robust description of the Content Connections at the high school level, see earlier in this chapter.

CC 1 Vignette: Whale Hunting

Course: MIC 1, Integrated Math I, Algebra I

Content Connection: Communicating Stories with Data

Driver of Investigation: Impact the Future (Affect)

Domains of Emphasis: HS.F-BF, HS.F-IF, HS.S-ID

SMPs: SMP.3, 4

Lesson Context: In the 1970s the stock (or number) of bowhead whales in the Bering Sea was calculated to be as low as 600–2000 whales, mostly due to heavy commercial whaling. This was, of course, mightily concerning to environmentalists and thus the International Whaling Commission completely halted permissions to hunt whales hoping to restore the population. Commercial whaling had long been a known issue, and it was already restricted, but this really hurt native populations that hunt bowhead whales for subsistence. Note that this provides a good opportunity for discussing and reinforcing California Environmental Principle I, “The continuation and health of individual human lives and of human communities and societies depend on the health of the natural systems that provide essential goods and ecosystem services.”

Included below is an example of the practice from the perspective of an indigenous person from the region:

> “Subsistence whaling is a way of life for the Inupiat and Siberian Yupik people who inhabit the Western and Northern coasts of Alaska. From Gambell to Kaktovik, the bowhead whale has been our central food resource and the center of our culture for millennia, and remains so today.
>
> Our whale harvest brings us an average of approximately 1.1M to 2M pounds of food per year (12–20 tons x 45–50 whales), which our whaling captains and crews share freely throughout our whaling communities and beyond to relatives and other members of Alaska’s native subsistence community in other native villages. For perspective, replacing this highly nutritious food with beef would cost our subsistence communities approximately $11M – $30M per year.
>
> As important as whale is to keeping our bodies healthy, this subsistence harvest also feeds our spirit. The entire community participates in the activities surrounding the subsistence bowhead whale harvest, ensuring that the traditions and skills of the past are carried on by future generations. Portions of each whale are saved for celebration at *Nalukataq* (the blanket toss or whaling feast), Thanksgiving, Christmas, and potlucks held during the year. […] Sharing the whale is both an honor and an obligation.”

Over the years, the International Whaling Commission (IWC) has worked with the Inupiat and Siberian Yupik people to ensure their needs are met and whales are protected. Through this process, bowhead whale populations have bounced

> back. However, the IWC still establishes whaling quotas for the local indigenous folks to ensure the population remains strong.
>
> The last ice-based abundance and Photo-ID-based surveys were conducted in 2011. The 2011 ice-based abundance estimate is 16,892 (within the range of 15,704–18,928). The rate of increase of the population, or trend, starting in 1979 was estimated to be 3.7 percent per year (within the range of 2.8–4.7 percent). These abundance and trend estimates show that the bowhead population is healthy and growing with a very low conservation risk under the current Aboriginal Subsistence Whaling management scheme." (IWC, n.d.; data from Givens et al., 2013)

Task: The tribe has assembled a committee of tribal scientists and community members, along with outside scientific and economic advisors, to make a recommendation to the International Whaling Commission. The proposal will specify how many whales the Inupiat and Siberian Yupik people will hunt this year as part of the Aboriginal Subsistence Whaling management plan, while making sure the whale population continues its growing trend. As a member of the committee, it is your task to help create the proposal.

The task as presented is deliberately very open-ended. Different student teams will consider many different factors (beyond Inupiat and Siberian Yupik hunting) that might affect the committee's recommendations and about which they might wonder—such as changing mortality rates due to shrinking ice cover, ship collision mortality, age structure of the population, etc. Described here is one student team's progression as they attempt to formulate a recommendation.

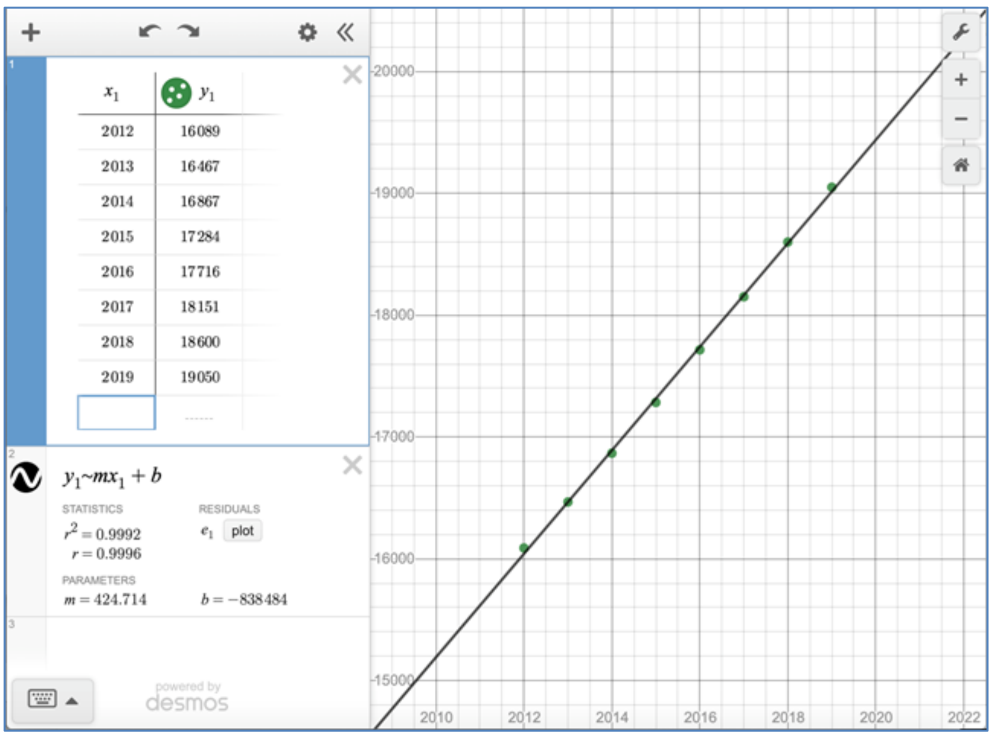
Student Vignette: The group receives the task, and discusses what they were being asked for. They decide to break down the problem into more manageable pieces, so they make a checklist with three items:

1. Figure out what happened to whale population between 2011 and 2019.

2. Find out the current growth rate that should be maintained.

3. Calculate how many whales can be lost in 2020 so that the growth rate is maintained.

For point 1, they think they might be able to find more data online, so they search statistics on whale hunting from 2011–2019. They found a table in the IWC website that lists every whale catch between 1986 and 2018. It contained more information than they needed: different whale species and stocks from different oceans, but they reviewed the information and pulled out the data they needed. In order to estimate the whale stock in 2018, for each year between 2011 and 2018 they plan to use the equation:



(Number of whales in the year they're looking for) = (Number of whales in the year prior)*(growth rate per year) – (whales hunted that year). This helped the students to use the growth over time to estimate the whale hunting in 2019.

They discuss with the whole group which numbers to use for growth rate and for the 2011 stock numbers, since they have the estimates but also the error ranges the experts gave. They decide that it's better to be safe than sorry, since whale overpopulation hardly seems like an issue, so they will use the lower end of the range for both numbers. Now comes a lot of number crunching, but computers can do that.

They use Wolfram|Alpha to quickly complete the calculations and they estimate the 2019 stock at 19,050.

However, they know they need the stock for the beginning of 2020. They don't have the data for how many whales were hunted in 2019, so they estimate it by averaging the years they do have data for: 2011–2018. The average is 60.75, so they round it to 61 and use their equation to calculate the stock at the beginning of 2020 as 19,522.

Now they look at point 2: finding the rate at which the population is currently growing. They use Desmos to graph the population each year and map a line of best fit, which will show the target growth rate.

That leads them to point 3: how many whales can be killed to keep this target? They look back at the original growth equation, but now they solve it for how many whales can be hunted:

(whales hunted that year) = (Number of whales in the year prior)*(growth rate per year) – (Number of whales in the year they're looking for)

- That target growth line has the equation y = 424.714 x – 838,484, so for x = 2021 (meaning, after the hunt in 2020), the population target would be 19,863, and they already know the growth rate they've been using, and their estimate for the 2020 population, so they can calculate the number of whales that can be hunted while maintaining the current growth and make a recommendation to the IWC.

  Note: This provides a good opportunity for discussing and reinforcing California Environmental Principle V, "Decisions affecting resources and natural systems are based on a wide range of considerations and decision-making processes." It demonstrates the importance of mathematical analysis in making policy recommendations and decisions about the conservation and management of organisms and the ecosystems they depend on. It also reinforces California Environmental Principle II, "The long-term functioning and health of terrestrial, freshwater, coastal and marine ecosystems are influenced by their relationships with human societies.

The team is finally tasked with preparing a presentation of their results to the rest of the class. This team presents their work in a slide presentation; another team prepares a website, and a third a poster.

CC 2 Vignette: Drone light show

Course: MIC3, Integrated Math III, Algebra II

Content Connection: 2 Exploring changing quantities

Driver of Investigation 3: Impacting the Future

Domains of Emphasis: HS.A-SSE, HS.A-CED, HS.F-BF, HS.F-TF, HS.G-GMD, HS.G-MG

SMPs: SMP.4, 5, 7

Source: Consortium for Mathematics and its Applications (COMAP), High School Mathematical Contest in Modeling (HiMCM)—2017 Problems.

Problem: Drone Clusters as Sky Light Displays

Intel© developed its Shooting Star TM drone and is using clusters of these drones for aerial light shows. In 2016, a cluster of 500 drones, controlled by a single laptop and one pilot, performed a beautifully choreographed light show.

Our large city has an annual festival and is considering adding an outdoor aerial light show. The Mayor has asked your team to investigate the idea of using drones to create three possible light displays.

Part I – For each display:

a) Determine the number of drones required and mathematically describe the initial location for each drone device that will result in the sky display (similar to a fireworks display) of a static image.

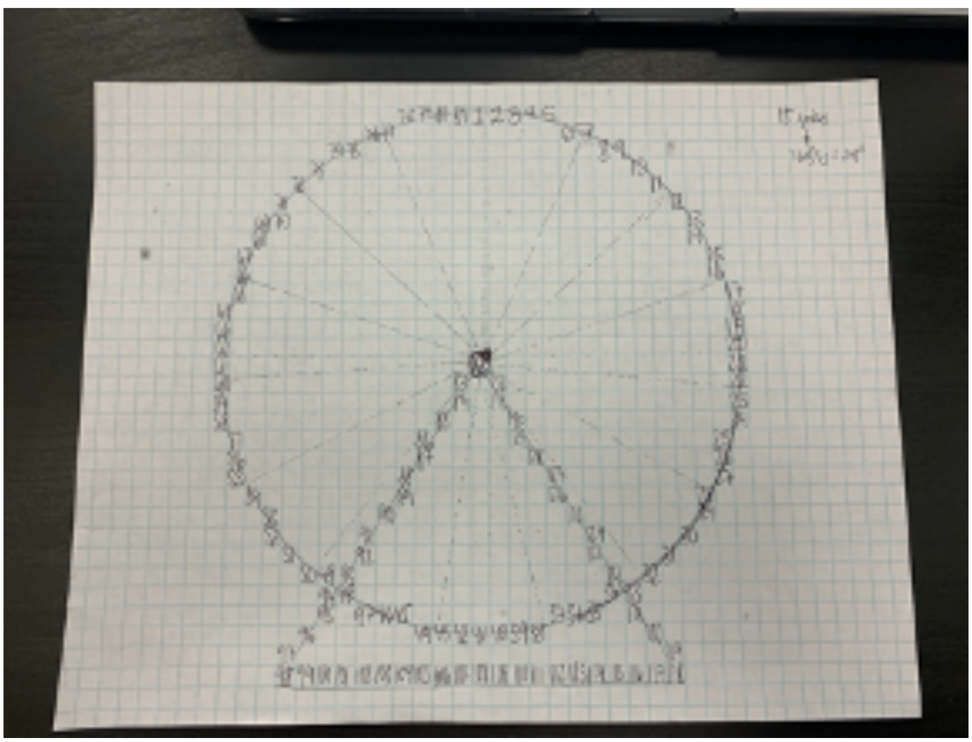
b) Determine the flight paths of each drone or set of drones that would animate your image and describe the animation. (Note that you do not have to actually write a program to animate the image, but you do need to mathematically describe the flight paths.)

Students are instructed to work together in three groups to design a solution to the problem. All three groups start out by reading the task and discuss the task. They are then given access to the video, which includes closed captioning, and then prompted to conduct a search for photos and clip art of Ferris wheels as a type of moving light system. Some groups want to watch the video several more times to be sure they understand. From experience, they know that this is not the kind of problem that allows them to find the answer in the back of the textbook. This kind of a problem can be approached in a variety of ways, and the challenge of the openness of the problem is thrilling! This flexibility aligns with the UDL principle - Provide multiple means of engagement by optimizing individual choice and autonomy. Students will need to think about the math tools and processes they have already learned before and apply them to a new context. This can be understood as the “formulate” stage of the Modeling Cycle.

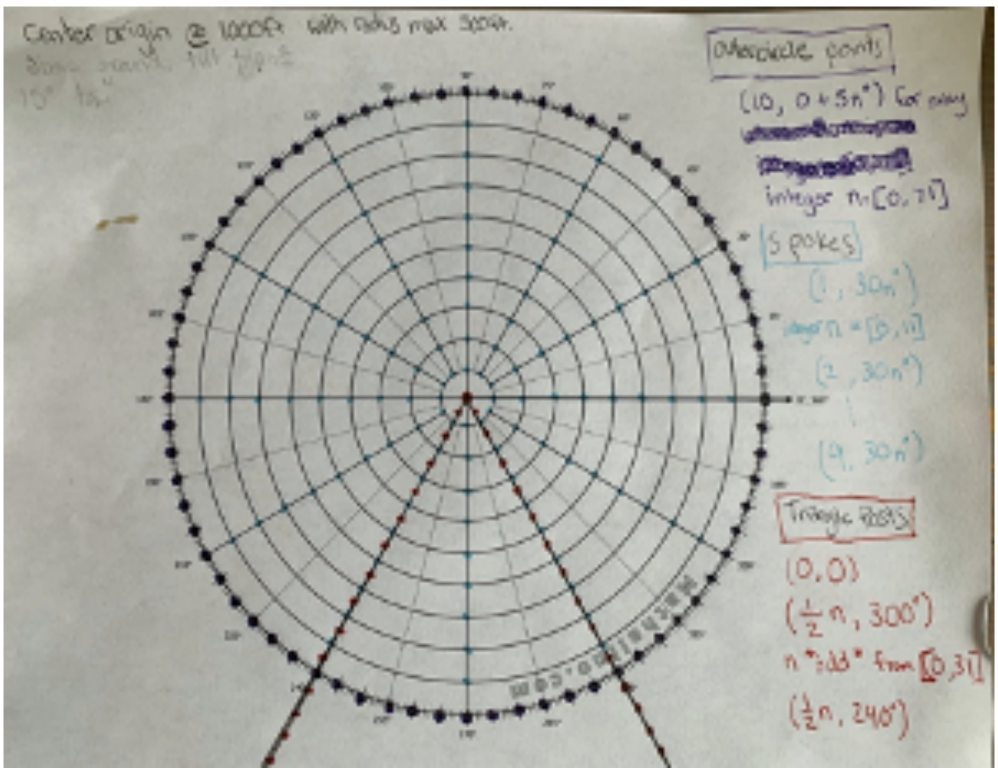
Over the course of the year, students have had several previous opportunities to engage in the math practice of modeling. Students know that math models help both to describe and predict real-world situations, and that models can be evaluated and improved. With every group member contributing to the brainstorm, students quickly start sketching as a way to visualize solution paths. As students are drawing, they explain and label their diagrams to show the “initial location,” for example. Some students are eager to get to display three, where they get to create their own design.

The teacher notices three unique approaches arising in the groups’ work, particularly in how they have decided to model the changing quantities within the problem. The teacher is pleased to see use of visuals and diagrams, as these are important ways of seeing and understanding mathematics and critical supports for students. As the teacher listens to the small group work, she acknowledges how well the groups are making space for everyone’s ideas. At first, the teacher notes that students are not writing much, but she has learned not to intervene too quickly. Instead, she allows their ideas to build, with the firm belief that her students will make progress.

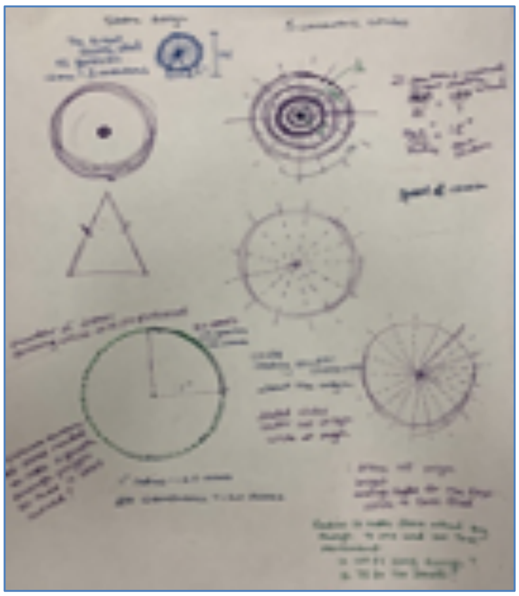
Group A: The students in this group have decided to model the problem on the idea of pixels in a grid that make up images on a television screen. The team draws an image of a Ferris wheel on the grid, and numbers every “pixel” in their grid that will need to be lit up by a drone to represent the circumference of the Ferris wheel. Next, the group has decided to model the rotation of the wheel by programming some drones to stay in place and some to move in a particular pattern. They know the pixels for the triangle don’t move so these drones will be programmed to stay in place. And for the circle, it’s a loop.

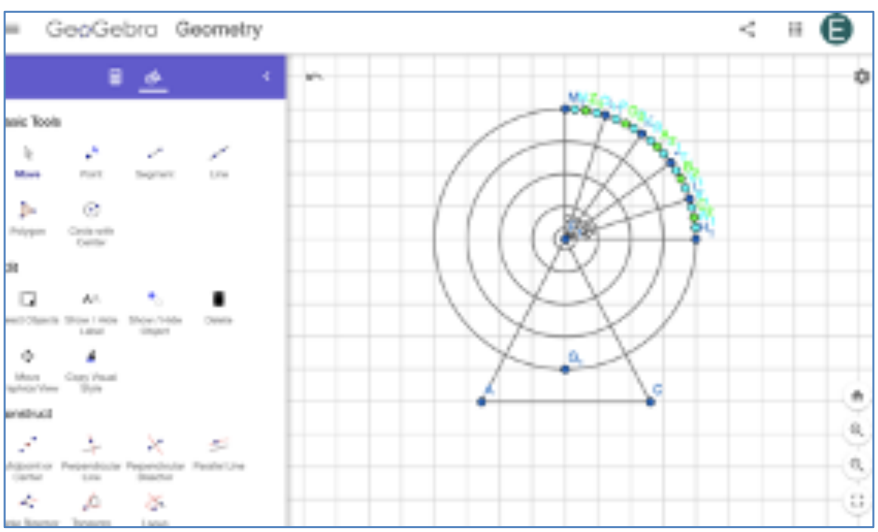


Group B: In this group, students have decided to model the Ferris wheel using polar coordinates. They decided that programming the coordinates (x,y) for the drones that make the circle of the Ferris wheel would require defining a unique x and y for every single drone! But, in polar coordinates (r,theta), the outer circle of the Ferris wheel can be thought of as many points in the plane sharing the same radius, which means that they would only need to change the theta for each drones coordinates and keep the r the same. The group determines with coordinates representing the wheel, spokes, and triangle posts of the Ferris wheel. To model the rotation of the wheel, the angle (theta) that each drone is programmed to will increase by 5° for a total of 72 moves of the circle to complete one full rotation of the wheel. To model the rotation of the spokes, the angle (theta) that each drone is programmed to will increase by 30° for a total of 12 moves, to complete one full rotation of the wheel. The drones placed to represent the base of the Ferris wheel are programmed to stay in place.



Group C: This group selected an image of the Great Seattle Wheel to use as their guide. They decided to model the image of the Ferris wheel using the equation of a circle in the cartesian plane, and various dilations of the outer circle to create inner circles that will model the spokes of the wheel. Finally, the group decides to utilize and online graphing tool that will allow them to rotate the image within the plane to model the turn of the wheel. The group creates equations for 20 lines that start at the center of the circle, intersect each concentric circle, and end at the outer circle. While this is a slight modification to the 21 spokes on the Great Seattle Wheel, it allows the degrees of each arc length to be integer values, which the students agree will be easier to work with. These lines separate the circle into 20 equal sectors—each with an arc length of 18°. They decide to program a drone at each intersection of the circles and the lines to represent the spokes. A discussion ensues about the number of drones that must be placed between each spoke intersection on the outer circle to create an outline of the circle that looks smooth, the group decides on three for now because 18° is easily divided into three. Ultimately, the group decides to utilize an online graphing tool (GeoGebra) that will allow them to rotate the image within the plane to model the turn of the wheel. The group discusses the rate of rotation and degree of rotation that would be most appropriate to model the movement and speed of the Great Seattle Wheel.





After students have worked out the details of their models, each group presents their approach to the problem. Some students jot a few notes down to help them remember key ideas and terms. They prepare to describe their model and explain their choices to their peers. Students prepare a poster, using colors to highlight key features of their model. The teacher circles around and helps students who want to do a quick run-through of their presentation, giving students feedback to strengthen their work, supporting language learning by clarifying how content vocabulary supports the mathematics, and suggesting ways to better convey the information in presentation-worthy academic discourse as she does so. Each presentation is followed by a short question and answer session. Each presentation poster is displayed at the front of the class, clearly showing a wide range of methods and approaches.

Following these presentations, the teacher conducts a Gallery Walk, allowing smaller groups of students to spend a few minutes viewing the posters up close. This activity is followed by a whole-class discussion on the different strategies taken by each group, including a discussion about the affordances and challenges presented by each choice for modeling the changing quantities in the problem. Throughout this process, the teacher is taking notes on feedback, including areas of strength and where possible improvement is needed as students engage with the modeling cycle. She will use this information in responding to the students' presentations during evaluation, and framing the next modeling task.

Disciplinary Language Development

This task provides extended opportunity to deepen in the area of mathematical modeling within an authentic context. The challenging nature of this task encourages collaboration, building on one another's ideas and key skills using students' mathematical language. In groups, students make use of the full array of mathematical resources to construct their models, utilizing prior mathematics learning. The visual nature of the task, along with the video, and their presentation posters expand the modalities in mathematics, supporting the guidelines in Universal Design for Learning

(UDL), which move beyond the more typical confined to calculations and symbols. Here, the visuals are not support for their models, they are the models themselves.

CC 3 Vignette: Blood Insulin levels

Grade level: MIC I/Integrated Math I/Algebra I

Content Connection 3: Taking Wholes Apart and Putting Parts Together

Driver of Investigation: Make Sense of the World (Understand and Explain)

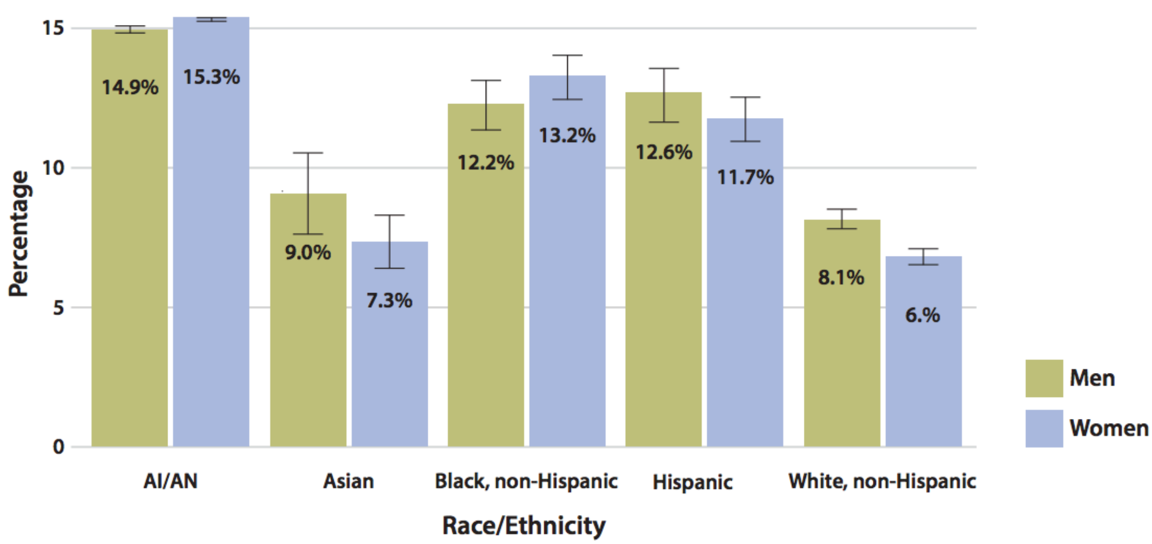
Domains of Emphasis: HSF.IF.A, HSF.IF.B, HSF.IF.C, HSF.LE.A, HSF.LE.B

SMPs: SMP.1, 4, 5

Ms. Alfie loved science and all things mathematics. She found that her Mathematics I students came to her from various backgrounds and experiences and they did not feel the same way she did about STEAM subjects. She was excited to teach Integrated Mathematics I using Core Plus with the goal of exciting her students about the role mathematics plays in the world around them.

Ms. Alfie was midway through the first year of IMI and felt her students were ready for a math investigation that included medicine, coming from Core Plus 1. In her materials she found several examples that included the concept of half-life and she wondered how she could use a medical context to introduce exponential functions. She also wondered how students would embrace the topic, knowing that fractions and number sense were not topics students felt confident about. The activities they had completed around linear functions earlier in the year had helped them learn to interpret slope as a fraction and interpreting slopes within the context of the problem. For example, Ms. Alfie's students were happy to consider an equation in the form $y = 3/4x + 5$ as starting at the y intercept, (0,5) and increasing 3/4 of a unit vertically for every horizontal step. They also thought about it as three steps up and four steps right for every unit. She wanted to challenge and extend her students' thinking about rates of change that were not constant, for example exponential decay in context, i.e., every 60-minute increase in time the amount of drug might decrease by 50 percent in the body.

Ms. Alfie began the unit by doing a graph talk, using real world data from the Centers for Disease Control (CDC). A graph talk is a math routine where students were asked to study the graph and be ready to share what they notice and wonder. Ms. Alfie purposefully left the title of the graph off and asked students to brainstorm what the data was about. This is analogous to students reading a news article and having to develop a "headline" that captures the main idea.



[Link to long description]

Source: Centers for Disease Control and Prevention, 2017.

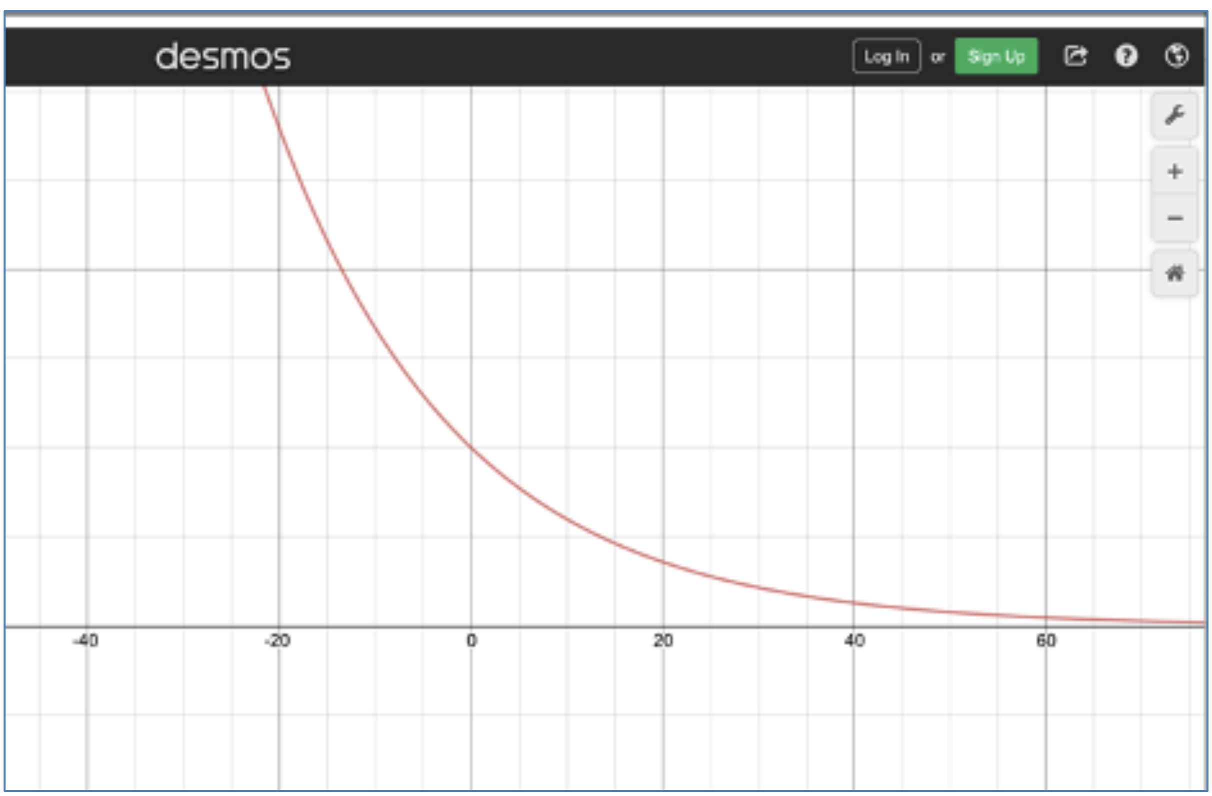
As students discussed the graph and the information they wondered if the graph showed participation in sports, academic clubs, or favorite television shows. Her students did not come close to the actual story (a way of creating a narrative to express what is being communicated) of the graph which shows data of the estimated age-adjusted prevalence of diagnosed diabetes cases in the US for adults from 2013–2015. But Ms. Alfie knows that with more experiences with interpreting graphs and other visual display of data, her students would learn to identify the main themes.

The activity was supported by Ms. Alfie’s collaboration with a teacher who supported content-specific English Language Development (ELD) instruction to English learners in her class. This designated ELD support included helping the students to understand and develop the critical language and grammatical structures necessary for successful engagement in this activity. With this base of understanding, Ms. Alfie’s lesson could focus on integrated ELD support and ensure all students had the access necessary to engage with the work.

The students were prepared when, after the data talk and the story reveal, Ms. Alfie asked the class to spend 20 minutes in small groups looking up information on diabetes. Each group had three types of roles: the recorder, the searcher/investigator, and brainstormers. Ms. Alfie was aware that for many students in the community, diabetes was not any medical condition, but one that affected family members deeply. She framed the investigation around using math and data science more specifically to understand the prevalence and treatments of diabetes. This was a mathematical investigation of a real-world problem, and it relied on scaffolding the context with specific medical vocabulary. On this language foundation, the first step in understanding a real-world phenomenon is to gather information. She asked each group to share the research they had found and as a class the discussion continued about the disease as well as the use of prescription drugs to improve the health and well-being of people living with the disease. Ms. Alfie then asked students to look for more information about diabetes and the hormone, insulin, and the role it plays in the body. Information was not just limited to online research. The community clinic also had pamphlets and health advice about diabetes. The students discussed the difference between public information (in the form of a pamphlet) can differ from online internet searches and sources. Ms. Alfie used these different texts to focus students as they looked closer at issues around the dosing of insulin, as it is a common therapy for diabetes.

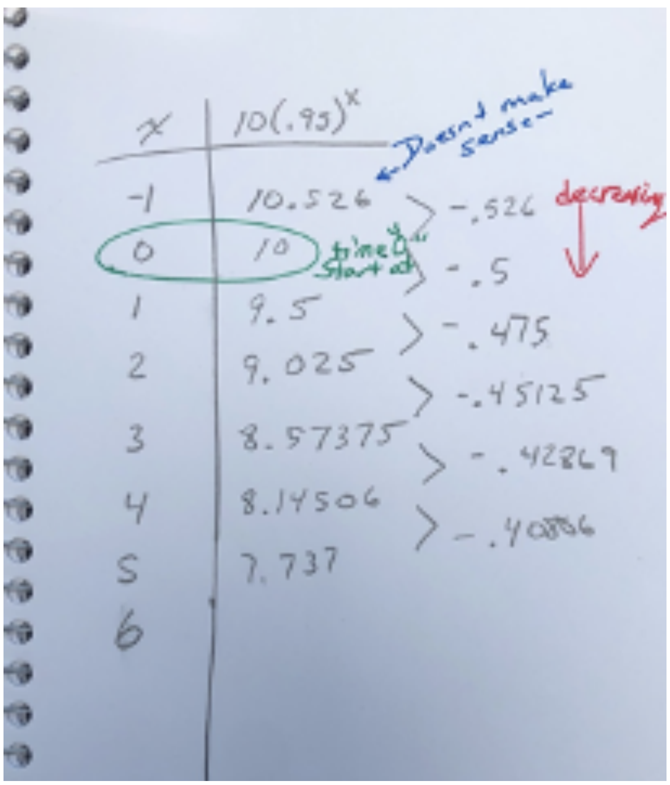
First Ms. Alfie shared with students the function: $y = 10(0.95)x$. She explained to students that the body metabolizes drugs in an interesting way and while different bodies process drugs differently we can model the metabolism of a drug with a function. Her multilingual students had worked with the science vocabulary in the lesson, and helped support her when other students needed support with understanding the meaning of "metabolize." Students looked up varying definitions and came to understand that it means to "break down" over time in this context. (Assess the multilingual students' understanding of phrasal verbs such as "break down" and "look up," and conduct a mini-lesson on these linguistic structures, if necessary.) And it turns out that different medicines break down at different rates in our bodies. Although it seems like a straight-forward definition, many students could possibly do all computations without ever understanding this central idea.

Ms. Alfie returned to the idea of representing data in the form of a story. She told students the equation told a story of insulin metabolism and she asked students to use DESMOS to illustrate and study the function. In groups, students were asked to study the graph and make a table of values where *x* represented time and y represented the units of insulin that were injected at t=0. Together, they brainstormed responses to the question: What story does the function illustrate? Or put another way, how does the function behave?



Students worked together graphing the function and thinking about what the values meant in the table as well as the values that were in the function. Students did not always agree on how to interpret the graph or the values of the function. When they disagreed, members took turns explaining their reasoning, and responding to questions from their peers. To explain more clearly and avoid unnecessary confusion, they decided to label their axes, agree on phrases such as, “When *x* is 20, *y* is [blank],” and so on. They discussed as a class how the function was decreasing and how the output was decreasing in a way that was not linear. This prompted a discussion of questions students generated, such as: What insulin level is too high or too low? What dosage is needed to maintain a safe level? And What happens when you skip a dose or delay for hours?

Figure 8.4



Ms. Alfie asked students to think using various forms of mathematical representations beyond graphs. She introduced the table in Figure 8.4 to stimulate more thinking.

She posed the following questions:

- What is the initial amount of insulin administered?
- How much time has passed when the amount of insulin is 50 percent?
- When does the amount of insulin reach zero?

As the lesson continued students asked questions about how often a drug should be administered and why some types of medicine say one time per day, two times per day and three times per day. The lesson continued with students analyzing different equations for drug metabolism such as penicillin, where the half-life is about 1.4 hours.

As a way of wrapping up the investigation, the teacher asked students to connect what they had learned about how insulin metabolizes in the body over time with the broader theme of diabetes awareness and treatment in the community. This reinforced the use of mathematics, as well as the terms and language acquired in the lesson, and helped students solidify their understanding. Some students still had lingering questions, such as: Do people have different metabolic rates? Why do some people take different dosages of insulin? Why do some take it at different times of the day? From the students' work and conversation, Ms. Alfie knew that the lesson had sparked solid mathematical thinking about variables. She wondered if a representative from the community health center could come speak with her class about these questions.

CC 4 Vignette: Finding the Volume of a Complex Shape

Course: Integrated II/MIC 2/MIC 3

Content Connection 4: Discovering Shape and Space

Driver of Investigation 1: Make Sense of the World (Understand and Explain)

Domains of Emphasis: HSN.Q.A, HSG.GMD.A, HSG.GMD.B, HSG.MG.A

SMPs: SMP.1, 2, 3, 5

Marina Lopez is preparing to teach her integrated high-school mathematics class 3, with a group-based interactive task that will help prepare students for learning calculus. She is using an approach that gives students the opportunity to explore a mathematics problem before being taught formal content that might help them solve it (Deslauriers et al., 2019). Her plan is to ask students to consider ways to find the volume of a complex shape, specifically a lemon. Prior to this activity, Marina has spent time in her class building and reinforcing group-work norms and she has previously made use of a structured approach to group work known as Complex Instruction (Cohen and Lotan, 2014) and specifically assigning roles for members of the groups. She continues to use this because of the ways it makes authentic use of different roles to reinforce the fact that students are important resources for each other.

She opens the task on the first day by asking students to discuss situations in which they might need to find the volume of a complex shape. Students consider packaging objects and the need to work out materials for packaging. Marina then shares that they will consider this in more depth by considering ways to find the volume of a lemon. She holds up a lemon and asks the class “How can we find the volume of a lemon?” While a few hands are immediately raised she does not call on anyone but tells the group they will have an opportunity over the next two days of class to answer the question using lemons and various resources. As students work in groups to tackle this problem, they will review what volume is and how it is measured, and how it relates to other measures of shapes such as surface area.

Marina knows that concrete materials are not just for elementary students. Mathematicians use models, illustrations, and visual representations to explore ideas, strategies that are highlighted in the guidelines of Universal Design for Learning (UDL). When students visualize they bring important brain pathways into their learning of mathematics. Prior to class Marina has setup a table at the back with different supplies including different colors of modeling clay, vases, knives, and cutting boards, pipe cleaners, scissors and a few other materials. Groups are free to choose from the assortment of materials provided. To facilitate the use of materials, students are instructed that only the resource manager is allowed to get up to get supplies from the resource table and they can only have three supplies out at one time. During the early weeks of her class Marina helped her class develop a set of group work norms and has previously used roles for groupwork so students are used to these structures and have been working on engaging productively in groups (see also Cabana, Shreve, and Woodbury, 2014). Note the image of the supply table in Figure 8.5 below.

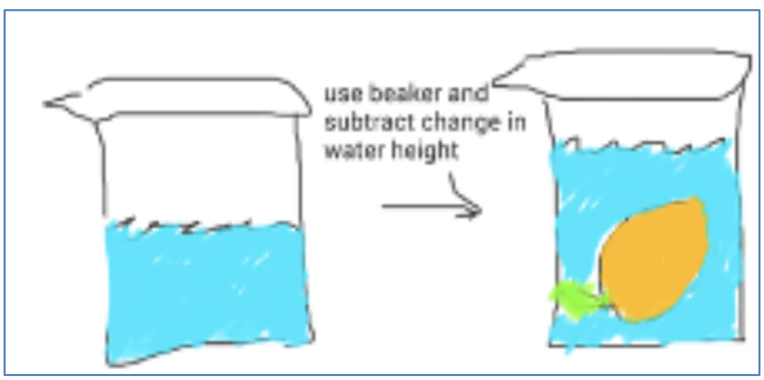
Figure 8.5

Animated noise begins to fill the room as students start talking in their groups and sharing their ideas. With much experience in group work, students exhaust the brainstorm process to collect as many ideas as possible and invite each group member to share their ideas. When ideas are not clear, they ask clarifying questions posted on the wall that promote justification and help students understand. Students also take one idea as a spark and build off it, elaborating and extending in new ways. Over time, these ideas become the group's ideas, not just the ideas from one person. They have been given one lemon for today but have also been told they will be able to get a second lemon tomorrow, so they have some freedom to play and even mess up their lemons.

As groups begin to dig into the problem, Marina reminds students to capture their ideas with notes, drawing, and sketches so that they don't lose track of their thinking. Students know not to worry about "complete sentences or perfect spelling" since they are just exploring ideas. Marina listens closely to discussion in each group, making quick notes of what she hears students saying. Their language is exploratory and imaginative at this stage of the lesson, e.g., “Would peeling the lemon help?” and “What about squeezing the lemon first?” and, “Is this a good way to cut it up?” Some of the students in class are multilingual and represent different levels of English language development. As designed, these students not only have access to the task, but also multiple opportunities to use language to explore their ideas and share their mathematical thinking. The concrete materials, small-group work, and structured group presentations all provide key supports in language developments.

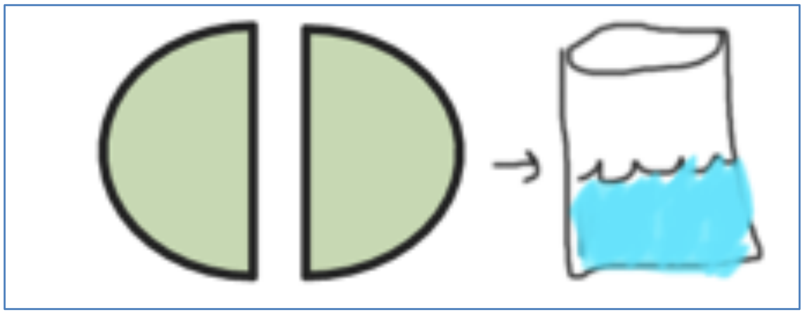
One group decided to use a bowl and water from the drinking fountain to see how the height of the water changes once the lemon is under the water. They draw a quick sketch to describe their idea (Figure 8.6 below). The students decide to use a marker to mark up the bowl like a beaker and begin filling it with water.

Figure 8.6



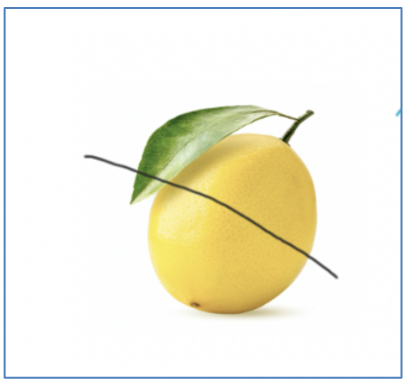
Another group has selected modeling clay and is attempting to make a mold of the lemon. They record their plan and describe that they will carefully fill the mold with water, and then find a way to measure the amount of water the mold holds (see Figure 8.7 below).

Figure 8.7



A third group has opted to use a knife and cutting board. They have decided that the shape of the lemon is very close to that of a sphere, so they can use the volume of a sphere formula to approximate the volume. To measure the lemons diameter and radius, they will cut the lemon in half, as shown in their diagram in Figure 8.8:

Figure 8.8

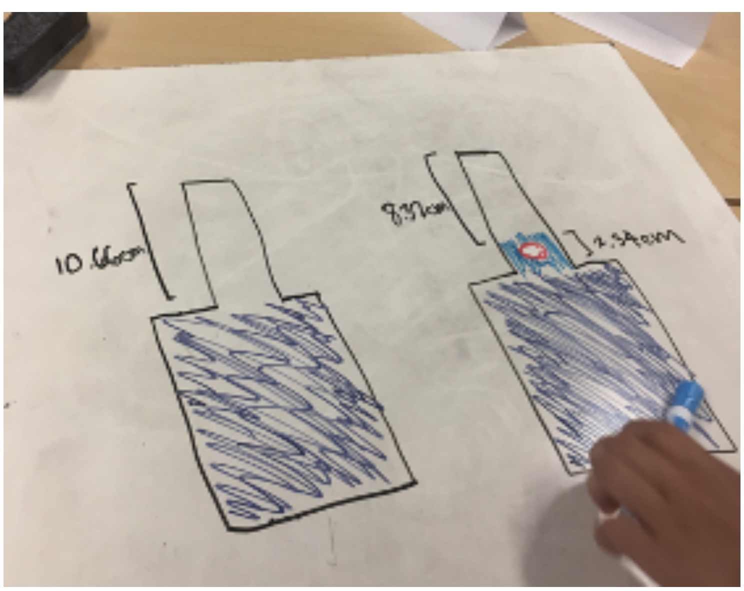


As this first period nears its end, Marina reminds students that they will be getting new lemons tomorrow so if they want to consider using the knives and cutting boards provided now would be the time. She also reminds them to be sure to document the work they did today and where they want to start tomorrow. They should plan to keep discussing and working as homework so they can be ready to create posters and present on day two.

For the second day of the project, students pick up where their work the previous day ended. One group finalizes its ideas and begins creating a poster to share their strategies with the class. Adam and Andres' group managed to try two ideas, but they engage in a debate over the best ways to present their work. Marina reminds her students that the group's reporter should take the lead in the creation of the poster, but that other roles in the group should be ready to share-out later in class. She says this as she walks among groups handing out additional lemons.

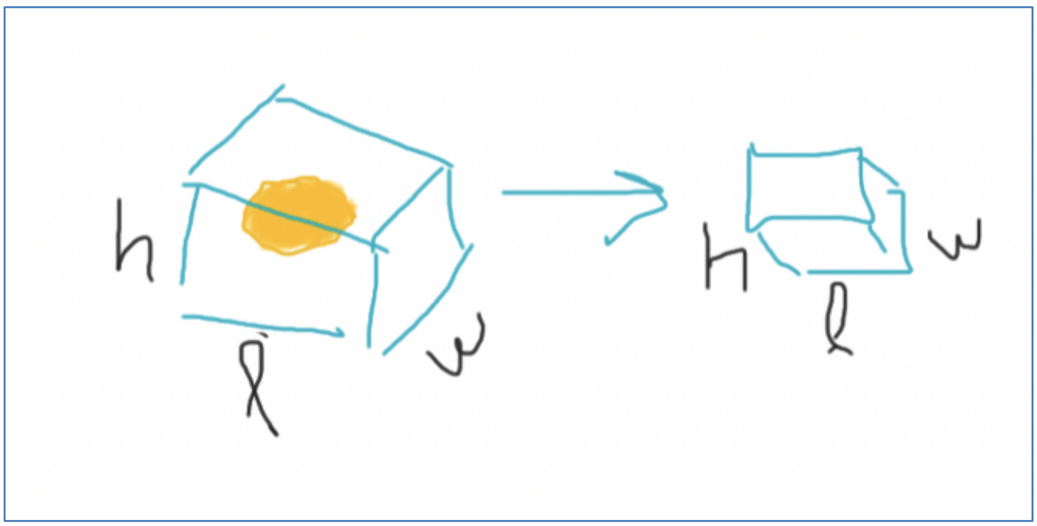
Marina knows that this is a group-worthy task because it draws on many aspects of mathematical thinking. Students are making connections to science and ideas of measurement through displacement, and to surface area, and still others groups are using a sort of "decomposition" approach by forming small cylinders. As she continues to circulate Marina, notes the different strategies she sees groups using to document their progress, and starts planning ways to sequence the group presentations so they meet specific learning targets she wants to highlight with this lesson.

After the 15 minutes pass, Marina calls her students back together and asks a group who attempted to use a water displacement method (but was not able to finish) to share first. As they share, she writes key phrases and words on the board that highlight their creative problem solving and calls on a second group that got further using a similar method. Marina asks this group to share their thinking and build on the work of the first group. Marina refers to her notes capturing what she heard during the groupwork as a way to highlight examples of mathematical language they were using. As this second group wraps up, Julio questions the group by wondering how the displacement method (shown below) might relate to his group's method of negative space.



Marina invites Julio’s group to present next. This group presents a solution using modeling clay surrounding the lemon and molded into the shape of a rectangular prism. First, they found the volume of their prism with the lemon inside, then they explained that they removed the lemon from the modeling clay and reformed it in the shape of a rectangular prism and found the volume again. They explained that the difference between the two volumes had to be the same as the volume of the lemon. Note their work in Figure 8.9 below.

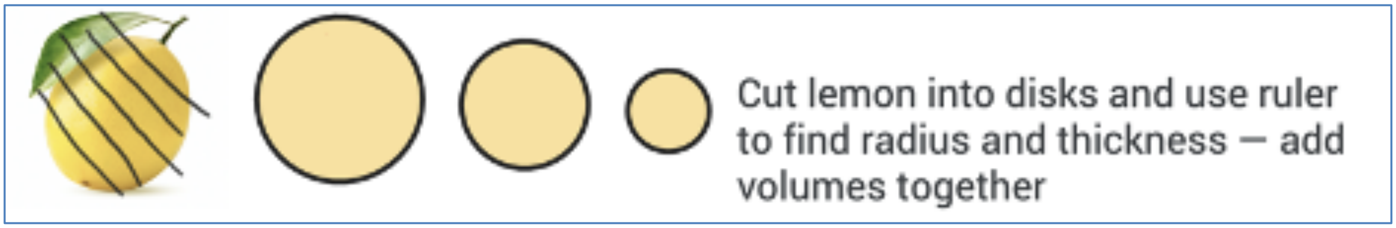
Figure 8.9



Other students in the class respond to this group’s idea with enthusiasm, citing excitement for its creativity. One student from the team that used a displacement approach raised her hand and connected with the idea that this team’s method was kind of like an “opposite” of what her team did. Several students nodded in agreement. The fact that students intuited the idea of “opposite” indicates that they paying attention to the relationship among methods, namely their inverse relationship which they cannot yet define completely. This is cognitively complex work which develops over time, and students are reaching into their mathematics to find words that convey their ideas.

Finally, Marina asks a fourth group to share their explanation. Silvia explains that the group tried many things, but their favorite method involved slicing up the lemon into many pieces. The group decided that each slice could be thought of like a very short cylinder. So, the group found the volume of each slice using the formula for the volume of a cylinder and then added them all together in the example below.

Figure 8.10



As Silvia explains her groups work, several other students appear to be taking notes and multiple hands are immediately raised to ask questions.

A whole class discussion ensues around the various strategies that groups utilized. Marina is careful not to rush the discussion, and to unpack students' comments and questions that she does not understand at first. At times, other students rephrase for one another to see if the idea is clearer. Marina poses the questions:

- "What are the strengths and challenges to these approaches?"
- "Which approach would you say is most accurate?"
- "How do you know?"

This metacognitive part of the lesson helps students move beyond just the lemon itself, towards noticing the methods they use in their analysis. The students take turns commenting on and comparing each other's strategies. Marina closes the class period by acknowledging the various mathematical practices that students engaged with and highlights the multiple dimensions of content that students utilized.

Long descriptions for Chapter 8

Figure 8.1: Content Connections, Mathematical Practices and Drivers of Investigation

Three Drivers of Investigation (DIs) provide the “why” of learning mathematics: Make Sense of the World (Understand and Explain); Predict What Could Happen (Predict); Impact the Future (Affect). The DIs overlay and pair with four categories of Content Connections (CCs), which provide the “what” mathematics (CA-CCSSM content standards) is to be learned in an activity: Communicating stories with data; Exploring changing quantities; Taking wholes apart, putting parts together; Discovering shape and space. The DIs work with the Standards for Mathematical Practice (the “how”) to propel the learning of the ideas and actions framed in the CCs in ways that are coherent, focused, and rigorous. The Standards for Mathematical Practice are: Make sense of problems and persevere in solving them; Reason abstractly and quantitatively; Construct viable arguments and critique the reasoning of others; Model with mathematics; Use appropriate tools strategically; Attend to precision; Look for and make use of structure; Look for and express regularity in repeated reasoning. Return to graphic.

Figure 8.3

Diagram indicating three pathways of courses indicating a variety of course offerings for Years 3 and 4 in high school. The preparatory courses are: Investigating and Connecting 1, Integrated I, and Algebra 1, followed by Investigating and Connecting 2, Integrated II, and Geometry. The later course options include: Mathematics: Investigating and Connecting: Functions and Modeling, Statistics, Calculus with Trigonometry, Pre-Calculus, Integrated III, Algebra II, MIC 3 as well as Other which indicates alternative mathematics courses not well represented in the other categories. Many possibilities exist for other courses, including financial mathematics, discrete mathematics, or further three-dimensional geometry explorations, for example. Return to graphic.

CDC Bar Graph

A bar graph includes data for age-adjusted estimated prevalence of diagnosed diabetes by race/ethnicity group and sex. The graph shows:

- American Indian/Alaskan Natives: men 14.9%, women 15.3%,

- Asian: men 9%, women 7.3%
- Black, non-Hispanic: men 12.2%, women 13.2%
- Hispanic: men 12.6%, women 11.7%
- White, non-Hispanic: men 8.1%, women 6%. Return to graph.