Mathematics Framework

Chapter 7 Mathematics: Investigating and Connecting, Grades Six through Eight

First Field Review Draft

Note to reader: The use of the non-binary, singular pronouns *they*, *them*, *their*, *theirs*, *themself*, and *themselves* in this framework is intentional.

Introduction

Grades 6–8 represent a critical period for teachers to direct students toward future success in high-school mathematics. Students' mathematics experiences in these grades significantly affect the likelihood that they persist in pathways that prepare them for the broadest range of options when they finish high school. Studies of students in grades 6–8 have found children to perceive mathematics as less valuable, and to report reduced effort and persistence in mathematics (Pajares & Graham, 1999). Students in these grades make choices about mathematics coursework “that will have long-term implications for their college and career achievements” (Falco, 2019). Girls in particular often exhibit a reduction in their sense of self-efficacy in mathematics during middle school (Falco, 2019), and self-efficacy is a significant predictor of success in high-school math (Petersen & Hyde, 2017). Students in groups that are underrepresented in Science, Technology, Engineering, Arts, and Mathematics (STEAM) fields experience significantly more academic barriers (lack of academic exposure) in middle grades and below, and these barriers are negatively associated with math achievement in high school (Williams et al., 2016).

(Sidebar) Authentic: An authentic activity or problem is one in which students investigate or struggle with situations or questions about which they actually wonder. Lesson design should be built to elicit that wondering. In contrast, an activity is inauthentic if students recognize it as a straightforward practice of recently-learned techniques or procedures, including the repackaging of standard exercises in forced “real-world” contexts. Mathematical patterns and puzzles can be more authentic than such “real-world” settings (from Chapter 1).

In grades six, seven, and eight, students build upon their understandings of concepts from grades TK–5, including place value, arithmetic operations, fractions, geometric shapes and properties, data and measurement. Although these topics form the mathematical foundations necessary for the transition to higher level mathematics, it is students’ curiosities about mathematics—and situations involving mathematics—that are critical to success in grades six through eight. In this regard, the National Council of Teachers of Mathematics (NCTM) calls for students to experience the “wonder, joy, and beauty of mathematics,” and for teachers to include the development of positive identity for all students in their learning of mathematics (NCTM, 2020). As is made clear throughout the framework, it is crucial in these grades to situate mathematics learning in situations that inspire authentic questions for students and increase the meaningfulness of mathematics in their lives.

In order to facilitate students' curiosity and positive disposition toward mathematics, teachers must look to provide an active learning environment filled with wonder and recognition of connections among the various topics, an environment which affirms for students that their learning is part of the magnificent and coherent body of mathematical understanding. Instruction should provide evidence that students' thoughts in and about mathematics matter; in every hard-won realization, subtle and creative explanation, deeper connection, or complex idea they produce, they convey their understanding as developing mathematicians. In this sense, teachers are champions of the cause, and facilitators of learning, rather than disseminators of rote information. Chapter 2 shows ways teachers can cultivate this type of environment for their students.

Integrated Mathematical Practices and Content Connections

What evidence demonstrates understanding, curiosity, and positive disposition? Teachers may note that a student can express an idea in their own words, can illustrate their thinking pictorially, can build a model using manipulatives, or can provide examples and possible counter examples. Educators may be able to observe that students are making connections among ideas or that they apply a strategy appropriately in another related situation (Davis, Edward 2006). Many useful indicators of deeper understanding are embedded in the Standards for Mathematical Practice (SMPs). Students show their ability to "analyze… the relationships in a problem so that they can understand the situation and identify possible ways to solve it," as described in SMP.1. Other examples of observable behaviors specified in the SMPs include students' abilities to perform the following tasks:

- use mathematical reasoning to justify their ideas (SMP.2,3)
- draw diagrams and other visual representations of important features and relationships (SMP.7)
- select tools that are appropriate for solving the particular problem at hand (SMP.5)

- accurately identify the symbols, units, and operations they use in solving problems (SMP.6)

Teaching mathematics for understanding requires active, intentional cultivation of students' use of the SMPs to ensure they develop the language of the discipline. As discussed in Chapter 2, considering the "big ideas" of mathematics and choosing investigations and tasks can allow students to learn big ideas and the many connections they offer. Big ideas are the central mathematical ideas that that link various mathematical understandings to a coherent whole. The cluster headings that organize the CA CCSSM represents one approach to big ideas, and they can guide discussions and selections of tasks. Instead of planning teaching around the small topics or methods set out in the standards, or the chapters of textbooks, teachers can plan to teach the "big ideas" of mathematics (Nasir et al, 2014). Lessons designed around big ideas facilitate the linking of one or more Content Connections with the SMPs, and with one of the Drivers of Investigation, as described in Chapter 1. (See for example big ideas across grades K–8: https://www.youcubed.org/wp-content/uploads/2017/11/Big-Ideas-paper-12.17.pdf.) As noted in Chapter 2, providing mathematics teachers with adequate release time collaborate with colleagues and engage in discussions around big ideas in their grade level or course can enable them to create rich, deep tasks that invite students to explore and grapple with those big ideas (c.f., Arbaugh, & Brown, 2005). More detail is given on big ideas in Chapter 2.

Guidance for ways to develop students' language proficiency as they learn mathematics is provided by the English Learner Success Forum (https://www.elsuccessforum.org/math-guidelines/math-area-of-focus-1), and include five Areas of Focus:

1. Interdependence of Mathematical Content, Practices, and Language;
2. Scaffolding and Supports for Simultaneous Development;
3. Mathematical Rigor Through Language;
4. Leveraging Students' Assets; and

5. Assessment of Mathematical Content, Practices, and Language.

The introductory content to the CA CCSSM is explicit on this point: *"The MP standards must be taught as carefully and practiced as intentionally as the Standards for Mathematical Content. Neither should be isolated from the other; effective mathematics instruction occurs when the two halves of the CA CCSSM come together as a powerful whole"* (CA CCSSM, p. 3).

Mathematics is considered by many to be a universal language, filled with concepts that are recognized in various countries and cultures throughout the world. As with any content-area vocabulary, all students should be seen as learning the language of mathematics. Support for students who are linguistically and culturally diverse learners of English, implemented through targeted instructional strategies aligned with the California English Language Development Standards (CA ELD Standards) integrated with mathematics instruction, also supports mathematical development for all, such as the principles and routines shown in this chapter.

Students who are English learners are most supported in learning the language of English and mathematics when they are given the opportunity to reason about mathematics through small group and whole class discussions, listen to other students, and connect with their ideas (Zwiers, 2018). Students who engage in mathematical conversations develop important languages—English and mathematics—simultaneously. As Zwiers points out, it is more productive to create engaging tasks that challenge students to use reasoning, than to isolate particular words or use sentence starters: "We don't want to put the cart of language before the horse of understanding" (2018, p. 10). Language development is supported when mathematical ideas are paired, either visually or physically, with verbalizations. Tasks that show or require visual thinking and that encourage discussion are ideal, and students can be encouraged to start group work by asking each other, "How do you see the idea? How do you think about this idea?" Students who regularly engage in the SMPs develop habits of mind that enable them to approach novel problems as well as routine procedural exercises, and to solve them with confidence, understanding, and accuracy. They also learn the language of mathematics that is woven into the mathematical practices, and can begin to think and speak as mathematicians.

Instruction should be designed to provide appropriate supports so that students at all levels of language development can engage deeply with the important mathematical ideas of the instruction (Walqui & van Lier, 2010). Principles and strategies for language development—especially important for students who are English learners, but valuable for all students—can be explored in Moschkovich, (2013), Zwiers et al. (2017), and Zwiers (2018), among many others. Instructional principles that enable this engagement include:

- Focus on students' mathematical reasoning, not accuracy in using language (Moschkovich, 2013)
- Support sense-making (Zwiers et al., 2017)
- Optimize output and cultivate conversation (Zwiers et al., 2017)
- Use student conversations to foster reasoning and its language (Zwiers, 2018)
- Maximize linguistic and cognitive meta-awareness (Zwiers et al., 2017)

Planning with deliberate instructional routines can support the implementation of these principles. The following list of recommendations references the CA ELD Standards that they help achieve:

1) Develop stronger and clearer ideas and language through iteration (Successive pair-shares; Convince yourself, a friend, a skeptic). CA ELD Standards Part I.A.1, Part I.B.5-6, Part I.C.9-12, Part II.B.3-5.
2) Collect and display student thinking and sense-making language (Gather and show student discourse; Number and Data talks) CA ELD Standards Part I.B.5-8.
3) Have students critique, correct, and clarify the work of others (Critique a partial or flawed response; Always-sometimes-never organizer to evaluate mathematical statements) CA ELD Standards Part I.A.1, Part I.B.5-6, Part I.C.9-12, Part II.B.3-5.
4) Create a need for students to communicate by distributing information within a group (Information gap cards and games) CA ELD Standards Part I.A.1-4, Part

I.B.5.

5) Allow students to explore a context and co-craft questions and problems (Co-craft questions; Co-craft problems). CA ELD Standards Part 1.A.1-4, Part I.B.5.
6) Create opportunities for students to reflect on the way mathematical questions are presented and equip them with tools to negotiate meaning (Three reads; Values/Units chart) CA ELD Standards Part II.5-8.
7) Foster students' meta-awareness and connection-making between approaches, representations, examples, and language (Compare and connect solution strategies; Which one doesn't belong?) CA ELD Standards Part I.A.1, Part I.B.5-6, Part I.C.9-12, Part II.B.3-5.
8) Support rich and inclusive discussions about mathematical ideas, representations, contexts, and strategies (Whole class discussion supports; Numbered heads together). (Zwiers et al., 2017) CA ELD Standards Part I.A.1-4.

Additional ideas related to these design principles are provided by Stanford University, Center for Language (SCALE), at https://ell.stanford.edu/sites/default/files/u6232/ULSCALE_ToA_Principles_MLRs__Final_v2.0_030217.pdf.

Videos of teachers and students working together on mathematics, with particular support for English learners, are available from the California Department of Education. An example of one such video is at https://www.youtube.com/watch?v=6wkPUcFO5JA.

We know that the term "English learners" masks a great deal of variability of experiences. For students at secondary level, researchers (Freeman and Freeman, 2002) have broadly considered groups of English learners who are newly arrived with adequate schooling, newly arrived with limited formal schooling, and long-term English learners. Newly arrived generally means less than four to five years. By understanding students' life and previous schooling experiences, schools can thoughtfully place students in the appropriate setting for learning mathematics. Older students who have had sufficient opportunity for schooling in their home language, are focused more on translating the content, building on their existing mathematics literacy skills. For students with limited formal schooling, programs serving these students in mathematics will need to provide rich experiences where students can develop academic literacy, perhaps even reading. For both sets of students, access to native language instruction serves as a bridge to mainstream English classrooms. Dual language and bilingual programs could offer mathematics courses in first language and in the target language, or some kind of blended approach.

Long-term English learners tend to present somewhat different characteristics from newly-arrived English learners. Many are U.S.-born and have attended U.S. schools for their academic history. They are native English speakers, thought it differs from their home language, and are placed in language development programs in their early grades because they speak a language other than English at home. A formal exit from most of these programs requires students to demonstrate English proficiency on state-approved assessments and (usually) also show on-grade level academic performance. As a high threshold, many students stay placed in language support programs for many years. Many of these students possess the ability to speak and write in conversational English, both with their peers and teachers, and are often indistinguishable from other English learners. These factors can mask the literacy needs they require to succeed academically. Long-term English learners tend to underperform, and are often placed in lower academic tracks. Knowing these particular characteristics of long-term English learners can help schools develop mathematics course pathways that are equitable, that provide the academic language scaffolding students need, but without grouping students with recent-arrival programs.

Across these broad groups of students, recently arrived with and without formal schooling and long-term English learners, the basic components of effective programs remain the same. Students should learn content with rich thematic instruction through attention to big ideas, with challenging and connected content, collaboration, feedback, language scaffolding, and respect for cultural diversity (Freeman and Freeman 2002).

Critical Areas of Instruction

In their TK–2 classes, students build a foundation for future mathematics as they explore numbers, algebraic thinking, operations, measurement, data, and shapes. They develop understanding of place value and use methods based on place value to add and subtract within 1,000. They develop efficient, reliable methods for addition and subtraction within 100. They work with equal groups and array models, preparing the way for understanding multiplication and algebraic thinking. They use standard units to measure lengths and describe attributes of geometric shapes and data. In grades 3–5 they extend their understanding of operations to include multiplication and division, and use of properties. Students in these grades also build an understanding of fractions, including equivalence and operations. They also acquire an initial understanding of measurement, both in terms of geometry as well as data science.

In grades 6–8, students deepen their understanding of fractions, especially division of fractions. This understanding also bridges to a new type of number—rational numbers—which are inclusive of all the number types previously seen (whole numbers, integers, fractions, and decimals). Students connect ratios, rates, and percentages, and use these ideas in engaging in proportional reasoning as they solve authentic problems. By writing, interpreting, and using expressions and equations, students can solve multi-step problem situations. Characterizing quantitative relationships using functions allows students to further develop understanding of rates and changing quantities. Measurement and classification ideas associated with two- and three-dimensional shapes and figures are connected to real-world and algebraic representations. Measurement questions extend to the need for measuring populations using statistical inferences with sampling.

It is important that teachers employ tools that can provide all students with access to the abundant, grade-level mathematics represented in the outline that follows. Rather than insisting on mastery of prior content—especially computational speed and recall—the goal should be access to the content of the current investigation. Thus, tools that allow increased focus on sense-making and building number sense (which might include calculators and online tools) should be readily available. These tools, when used strategically to inform ongoing planning and instruction, allow students to have greater access. Furthermore, the tools can be used to support students completing the California Assessment of Student Performance and Progress (CAASPP) assessments in mathematics.

These tools do not replace the fact that reinforcement or continued development of earlier grade-level understanding should not be included; but it should develop from engagement in grade-level investigations, not be a remedial precursor to such investigations. In other words, students' grade-level activities should develop a need for understanding previously-encountered ideas, so that students are ready to deepen that understanding.

When students show that they have gaps in their understanding or unfinished learning from previous grades, it is important for teachers to provide support without making premature determinations that students are low achievers, requiring interventions, or need to be placed in a group learning different grade-level standards. A helpful guide to intervention for times when students need support is given in Table 7X below:

Table 7X.

Common Misstep	Recommendation
Blindly adhering to a pacing guide calendar	**Use formative data** to gauge student understanding and inform pacing
Halting instruction for a broad review	**Provide Just-in-time** support within each unit or during intervention
Trying to address every gap a student has	**Prioritize most essential prerequisite** skills and understanding for upcoming content

Trying to build from the ground up or going back too far in the learning progression	**Trace the learning progression, diagnose, and go back just enough** to provide access to grade-level material
Re-teaching students using previously failed methods and strategies	**Provide a new experience** for students to re-engage, where appropriate
Disconnecting intervention from content students are learning in math class	**Connect learning experiences** in intervention and universal instruction
Choosing content for intervention based solely on students' weakest areas	**Focus on major work** clusters from current or previous grades as it relates to upcoming content
Teaching all standards in intervention in a step-by-step, procedural way	**Consider the Aspect of Rigor** called for in the standards when designing and choosing tasks, activities, or learning experiences
Over-reliance on computer programs in intervention	**Facilitate rich learning experiences** for students to complete unfinished learning from previous or current grade

Source: https://achievethecore.org/aligned/select-math-intervention-content/.

Students develop at different times and at different rates, and what educators perceive as an apparent lack of understanding may not indicate an actual lack of understanding. When students can benefit from extra support, it is advisable to offer them approaches that are different from the ways they may have previously been exposed to mathematics to ensure they have opportunities to engage in new ways.

Chapter Organization

The *Mathematics Framework* addresses the challenge posed by the principle of coherence through the shifts of big ideas, progressions of learning across grades (thus, grade-band chapters rather than individual grade chapters), and relevance to students' lives. A big idea becomes big when it includes connected mathematical content and practices and a driver for investigation—it is the combination of content, practices and investigation that makes content meaningful and important.

The four content connections described in the framework organize content and provide mathematical coherence through the grades:

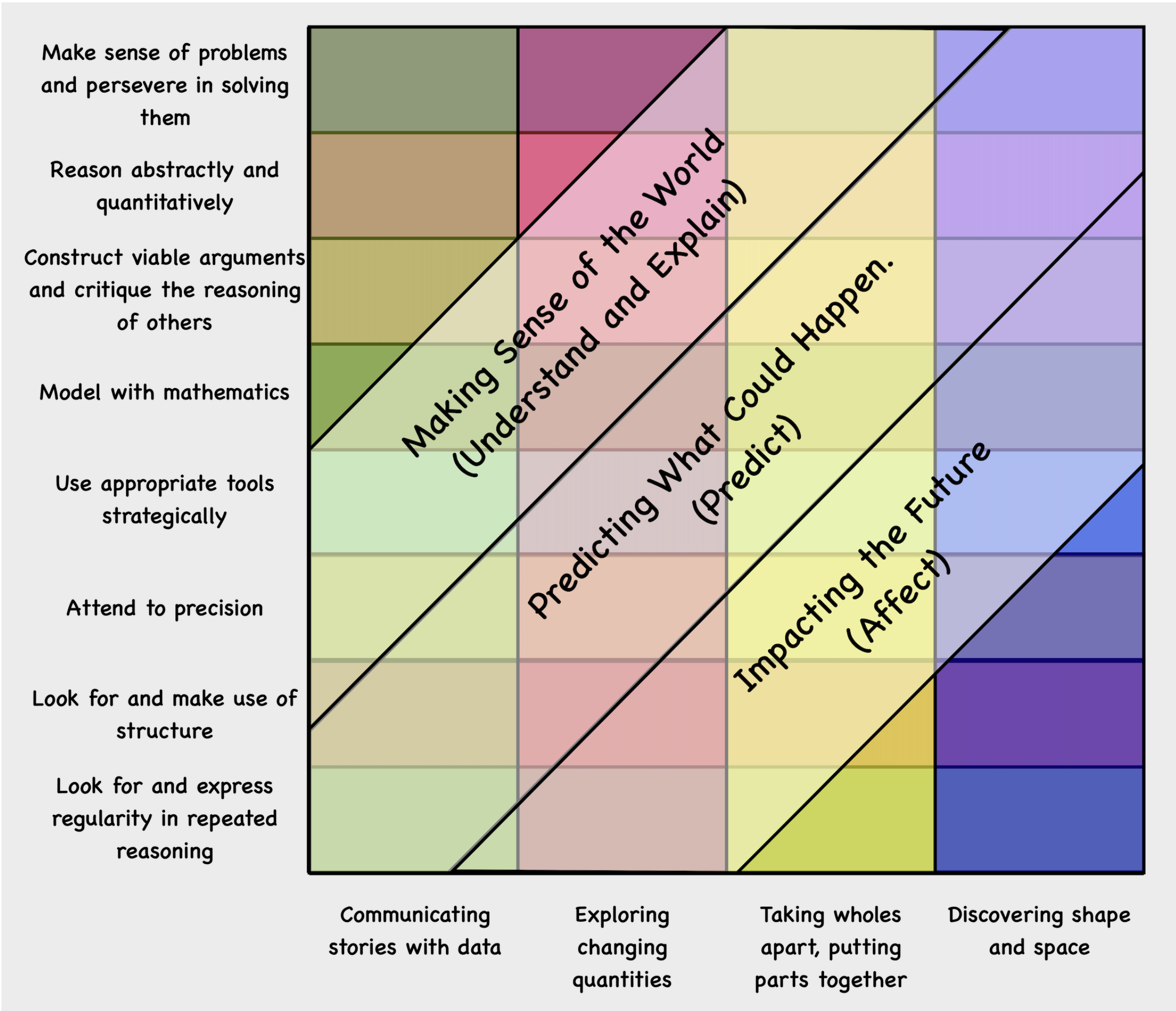
- (CC1) Communicating Stories with Data
- (CC2) Illuminating Changing Quantities
- (CC3) Taking Wholes Apart, Putting Parts Together
- (CC4) Exploring Shape and Space

These content connections should be developed through investigation of questions in authentic contexts; these investigations will naturally fall into one or more of these Drivers of Investigation:

- DI1: Making Sense of the World (Understand and Explain)
- DI2: Predicting What Could Happen (Predict)
- DI 3: Impacting the Future (Affect)

Big ideas that drive design of instructional activities will link one or more Content Connections, and SMPs, with a Driver of Investigation, so that students can Communicate Stories with Data in order to Predict What Could Happen, or Illuminate Changing Quantities in order to Impact the Future. The aim of the drivers of investigation is to ensure that there is always a reason to care about mathematical work—and that investigations allow students to make sense, predict, and/or affect the world. The following diagram is meant to illustrate the ways that the drivers of investigation relate to content connections and practices, as cross cutting themes. Any driver of investigation could go with any set of content and practices:

Figure 7X: Content connections, Mathematical Practices and Drivers of Investigation



Long description: Three Drivers of Investigation (DIs) provide the “why” of learning mathematics: Making Sense of the World (Understand and Explain); Predicting What Could Happen (Predict); and Impacting the Future (Affect). The DIs overlay and pair with four categories of Content Connections (CCs), which provide the “how and what” mathematics (CA-CCSSM) is to be learned in an activity: Communicating stories with data; Exploring changing quantities; Taking wholes apart, putting parts together; and Discovering shape and space. The DIs work with the Standards for Mathematical Practice to propel the learning of the ideas and actions framed in the CCs in ways that are coherent, focused, and rigorous. The Standards for Mathematical Practice are: Make sense of problems and persevere in solving them; Reason abstractly and quantitatively; Construct viable arguments and critique the reasoning of others; Model with mathematics; Use appropriate tools strategically; Attend to precision; Look for and make use of structure; Look for and express regularity in repeated reasoning.

Instructional materials should primarily involve tasks that invite students to make sense of these big ideas, wonder in authentic contexts, and seek mathematical investigation. As students discuss mathematical ideas, misconceptions or partial conceptions may arise that provide for rich discussion. If teachers work through investigations before they students embark on the learning, they can prepare them for such moments, and also note the ways mathematical practices emerge in the investigations. As students work through mathematical investigations, teachers and other students can engage in discussions around the ideas—the concepts and connections students come to are more important than answer finding, (see also Daro, 2013: https://vimeo.com/79916037). Big ideas in mathematics are central to the learning of mathematics. They link numerous mathematical understandings into a coherent whole and provide focal points for students' investigations. An authentic activity or problem is one in which students investigate or struggle with situations or questions about which they actually wonder. Lesson design should be built to elicit that wondering.

Tracking and Acceleration

Many California schools and districts struggle with questions of how and when to offer different pathways through K–12 mathematics. Recent attempts to address this issue include:

- Informal tracking beginning in upper-elementary grades
- "Honors" tracks beginning in middle school
- Various acceleration pathways, usually beginning in middle school, which either compress two years into one or have students skip a grade
- Ability grouping within classrooms
- Acceleration for all, e.g., California's eighth-grade algebra for all mandate beginning in 2008, with its overall negative impact on student achievement (Domina, McEachin, Penner, & Penner, 2015)
- High school pathways which diverge in grades 9, 10, or 11

Most of these approaches prior to high school are based on the premise that the only way to provide appropriate challenge for high-achieving mathematics students is to accelerate the pace of content learning. The CA CCSSM articulate a different vision of depth, coherence, and mastery, articulating progressions of learning that replace notions that teaching mathematics should be a race to abstraction and formality.

Mathematics is the most tracked subject in U.S. schools: approximately three quarters of U.S. grade-eight students are tracked in mathematics—a proportion that has not changed in many years (Loveless, 2013). Tracking, which is the term for the school-level practice of assigning students to more or less advanced courses in a subject, reflects a long history of inequity. Research has shown the inequity-reinforcing effects of tracking (e.g. Darling-Hammond, 2001; Callahan, Humphries, & Buontempo, 2020; Boaler & Staples, 2008; Boaler, 2014). A related practice, more common at elementary schools but also occurring in grades 6–8, is that of "ability grouping" within a classroom.

Two longitudinal studies, one in the United Kingdom and one in the U.S., followed students over three and four years, respectively, from the ages of 11 to 18. The studies aimed to consider the impact of tracking, curriculum choices, and teaching. In both studies, students in schools using mixed achievement groups achieved at significantly higher levels than students in schools employing tracking. In both cases, the schools using heterogeneous grouping did so as part of equitable initiatives and in both cases the schools using heterogeneous grouping reduced inequities during the time students were in school. The schools achieved success with heterogeneous grouping by using low floor high ceiling tasks that all students could access and that students could take to very high levels (see Chapter 2) and by having high expectations for all students. This success held across different countries, cultures and schools (Boaler, 2011, 2015, 2016; Boaler & Staples, 2008). Professional development designed to help teachers understand the value of low-floor/high-ceiling tasks and ways to teach with such tasks could embolden approaches to support students at different achievement levels.

When the teacher- and student-shared goal of mathematics class is the ability to produce correct answers to standard problem types, the teaching is easier in classrooms that are relatively homogeneous with respect to students' current speed in learning to replicate procedures. However, this one-dimensional sorting of students ignores the wealth and variety of assets that students bring to the classroom—assets which must be tapped in order to create the most powerful learning environments.

As a result of the documented issues with both tracking and ability grouping, the National Council of Teachers of Mathematics (NCTM) strongly advocates for creating a middle school mathematics that will "dismantle inequitable structures, including tracking teachers as well as the practice of ability grouping and tracking students into qualitatively different courses" (NCTM, 2020). The largest association of mathematics teachers in the nation, NCTM has made clear that districts and schools must confront the structural inequities of tracking and ability grouping, and to strengthen their efforts to support all students in learning along a common pathway.

In this framework, the vision of teaching and learning is that "students are actively engaged—developing mathematical curiosity, asking their own questions, reasoning with others, and encountering mathematical ideas in multi-dimensional ways" (Chapter 2). In the teaching environments this framework seeks to create and support, the varied strengths brought by a diverse student body are seen as advantages. Engaging in rich mathematical conversations can support students who are in the process of acquiring English by providing important opportunities to learn the language of mathematics and English. The pedagogy needed to develop powerful mathematics thinking for all students requires that they ask a variety of questions, demand clear—perhaps multiple—justifications from each other, and bring different cultural, linguistic, and representational backgrounds to the table. A supportive classroom environment in which language learners are learning reasoning through classroom discussions is ideal (Zwiers, 2018).

In a de-tracking initiative, New York City's school districts stopped teaching "regular" or "advanced" classes in middle school, and instead provided all students with content it previously labeled as "advanced." Researchers surveyed students in six cohorts for three years. The cohorts included three working in tracks and three following years when students worked in heterogeneous classes. The researchers found that the students who worked without advanced classes took more advanced math, enjoyed math more, and passed the state test in New York a year earlier than students in tracks. Further, researchers showed that the advantages came across the achievement spectrum for low and high achieving students (Burris, Heubert, & Levin, 2006). Similarly, eight California Bay Area school districts de-tracked middle school mathematics and gave professional development to the teachers. When they removed advanced classes and the majority of students took mathematics together, achievement increased significantly, with the untracked cohort 15 months ahead in mathematics. The de-tracking particularly helped high-achieving students (Boaler & Foster, 2018).

Historically, much of the acceleration and tracking that occurs during the middle-school years is due to a curriculum that is “unfocused, repetitive, and unchallenging” (Cogan & Schmidt, 1999). The CA CCSSM have removed much of the repetition, and the emphasis on progressions of learning in this framework is an attempt to make clear the ways in which mathematical expectations in each grade build on and extend those in earlier grades. For many adults (parents and educators)—who experienced a repetitive curriculum in which little important content learning was lost by skipping a grade—it can be difficult to see the dangers of many of the approaches listed above.

In grade eight, the CA CCSSM are significantly more rigorous than those from previous grade-eight content standards. They address the foundations of algebra by including content that was previously part of the Algebra I course—such as more in-depth study of linear relationships and equations, a more formal treatment of functions, and the exploration of irrational numbers. For example, by the end of the CA CCSSM for grade eight, students will have applied graphical and algebraic methods to analyze and solve systems of linear equations in two variables. The CA CCSSM for grade eight also include geometry standards that relate graphing to algebra in a way that was not explored previously. Additionally, the statistics presented in the CA CCSSM for grade eight are more sophisticated than those previously included in middle school and connect linear relations with the representation of bivariate data (see Chapter 5 for more details).

In 2012, the Noyce Foundation studied student placement in nine school districts in the San Francisco Bay Area and found that over 60 percent of students who passed algebra in eighth grade and/or met or exceeded state standards on the California Standards Tests were again placed in an algebra course when they entered high school, meaning they repeated the class they already passed. The study identified Black and Latinx students as much more likely to repeat and/or be placed in a low-level course. This effectively pushed students into a pattern of low achievement from which many never recovered, and the racially disproportionate placement outcomes led to legal liabilities for schools and districts (Lawyers' Committee for Civil Rights of the San Francisco Bay Area, 2013). This led to the California Mathematics Placement Act of 2015, which requires that districts and schools adopt a "fair, objective, and transparent mathematics placement policy."

The CA CCSSM Mathematics I and Algebra I courses build on the CA CCSSM for grade eight and are therefore more advanced than the previous courses. Because many of the topics included in the former Algebra I course are in the CA CCSSM for grade eight, the Mathematics I and Algebra I courses typically start in ninth grade with more advanced topics and include more in-depth work with linear functions and exponential functions and relationships, and they go beyond the previous high school standards for statistics. Mathematics I builds directly on the CA CCSSM for grade eight, and provides a seamless transition of content through an integrated curriculum.

The rigor of the CA CCSSM for grade eight means the course sequencing needs to be calibrated to ensure students are able to productively engage with the additional content. Specifically, students who previously may have been able to succeed in an Algebra I course in eighth grade may find the new CA CCSSM for grade-eight content significantly more difficult. The CA CCSSM provides for strengthened conceptual understanding by encouraging students—even strong mathematics students—to take the grade eight CA CCSSM course instead of skipping ahead to Algebra I or Mathematics I in grade eight.

The inequity of mathematics tracking in California can be undone through a coordinated approach in grades 6–12 (with attention to informal tracking beginning in elementary school as well). Unfortunately, many students, parents, and teachers encourage acceleration beginning in grade eight (or sooner) because of mistaken beliefs that Calculus is an important high school goal, and as such, Algebra I must be taken in eighth grade to ensure the student remains on track to reach a calculus class in grade twelve. See Chapter 8 for a full discussion of high-school pathways and the role of calculus in the high-school curriculum.

In summary, middle-school students are best served in heterogeneous classes that maintain appropriate challenge and engagement, and build depth of understanding, through meaningful mathematical tasks—as described throughout this framework. Skipping grades, or attempting to move grade six content to grade five or below, is not consistent with the CA CCSSM, and undermines the carefully-sequenced progression of topics they provide through the grade levels.

Students who are achieving below grade level—even those who are achieving at or above grade level—can be helped by the integration of the principles of Universal Design for Learning (UDL)—particularly learning mathematics through multiple forms of engagement, representation, action and expression (see http://udlguidelines.cast.org/?utm_source=castsite&lutm_medium=web&utm_campaign=none&utm_content=aboutudl). These guidelines set out an approach that describes an ideal mathematics approach, supported also by evidence from the neuroscience of the brain and mathematics learning (Menon, 2015).

Using Drivers of Investigation to Design for Coherence

As in all grade spans, grades 6–8 mathematics should be developed through activities that require students to build connections between mathematical ideas, and that are situated in authentic contexts that give students opportunities to generate questions. The Drivers for Investigation (DIs) outlined here are one organizing principle that emphasizes the need to begin instructional planning (and design of instructional materials) from big ideas that connect different areas of important mathematics, and that connect mathematics to students' lives, and make complex subjects easier to understand by integrating with other disciplines.

Drivers of Investigation

- DI1: Making Sense of the World (Understand and Explain)
- DI2: Predicting What Could Happen (Predict)
- DI3: Impacting the Future (Affect)

The DIs are in alignment with NCTM's recommendations in *Catalyzing Change in Middle School Mathematics* (2020) in that they broaden the purpose of learning to include curiosity, beauty and positive attitude toward mathematics; promote equitable teaching practices by focusing on students' agency; and develop deep understanding that unifies topics into meaningful investigations.

Using these DIs as an organizing principle for designing instruction (or instructional materials) can create an environment where student work will have a purpose—one where the work is driven by understanding, prediction, or change. Some content connections and mathematical practices may occur within multiple Drivers of Investigation—many of which may contain data, for example.

An example investigation for a DI3: Impacting the Future, is a task in which students learn the real-life story of a swimmer who is followed by a baby whale. The swimmer needs to decide if she should continue swimming to the shore, possibly breaching or endangering the baby whale, or swim out to try and find the whale's mother. The students decide on a path of action that accounts for ways it will impact the future.

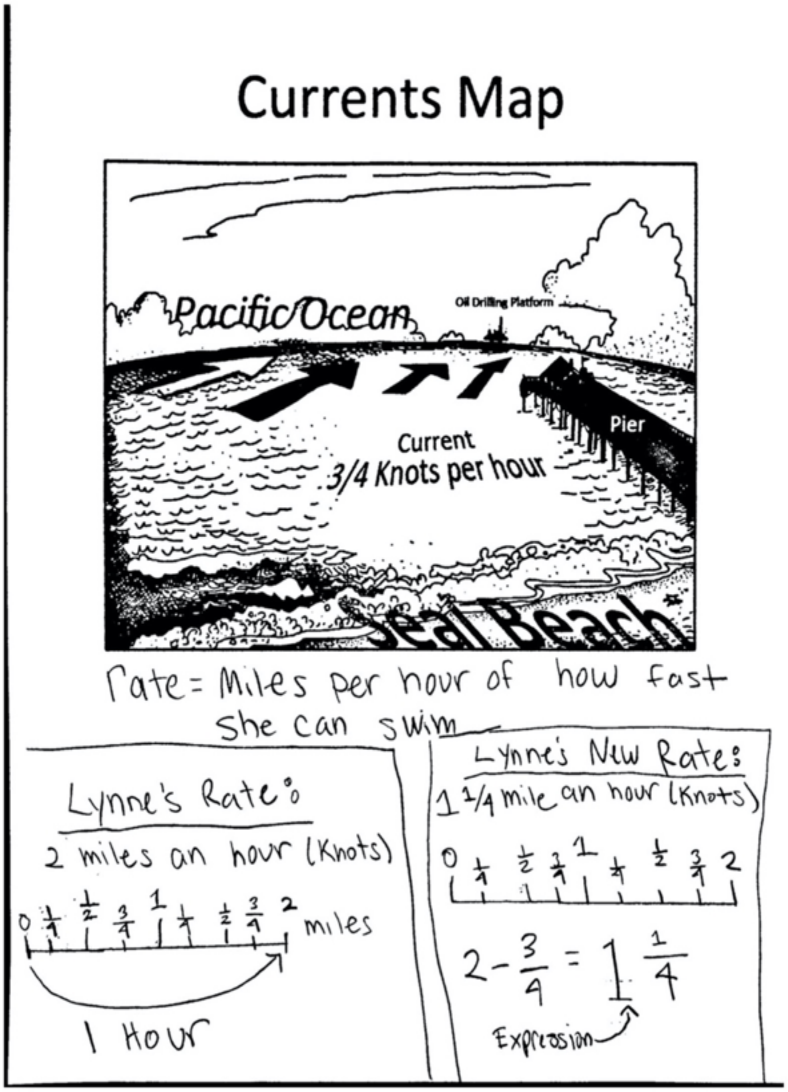
Vignette: Followed by a Whale

Whale beaching is an issue around the globe, and California is not immune. Whales need deep ocean water to live; if they swim too close to the shore, in shallow waters, they can be beached and die. Scientists are not sure why whales breach, but one possibility is that whales are very sociable animals and may follow another animal, especially one that needs help, into shallow waters.

The vignette below tells the true story of a class of fifth-grade students who worked on the whale story; but the content of the investigation can be oriented to middle school—with much of the content, especially linear relationships and proportions, appearing in the CA CCSSM for eighth grade.

Heather Herd read to her class the book *Grayson*, by Lynne Cox, recalling a true-life event of a 17-year-old swimmer who helped a baby whale. When Heather read the book, she saw an opportunity to engage her students in a powerful investigation influenced by mathematical problem solving. The unit she developed is appropriate for many grade levels, as it draws from mathematics in grades 5–8. The full lesson and video of Heather's class can be seen at https://www.youcubed.org/resources/data-science-online-course-lessons/.

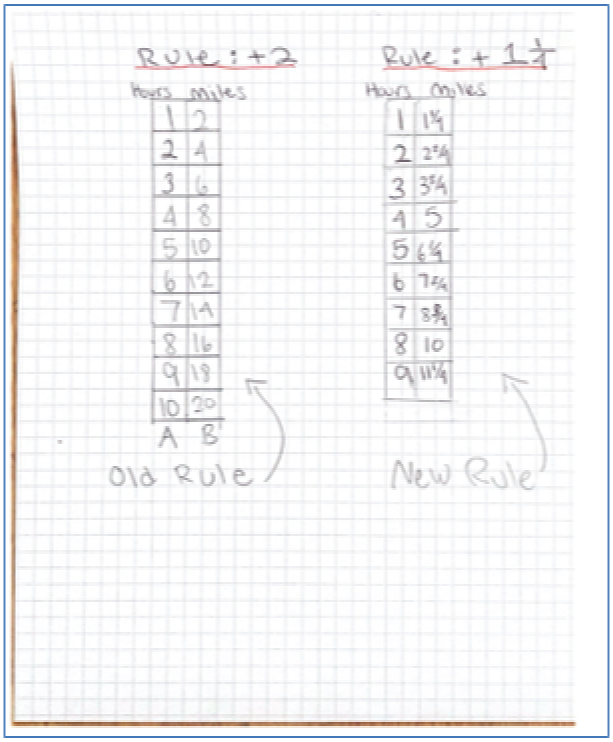
Figure 7X:

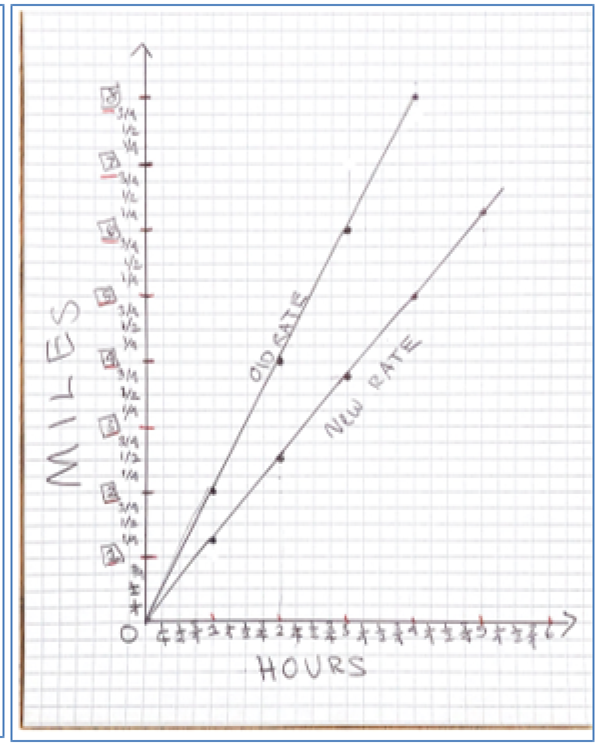


The story is set in the Pacific Ocean. At 17-years old, Lynne completed a three-hour swim workout in 55-degree water when she discovered that a baby gray whale had been following her. When she learned that fisherman had spotted a mother whale at a nearby offshore oil rig, which prompted a question: Should Lynne swim out to the oil rig with the baby whale, or should she swim to shore inducing the baby to follow her and possibly be in danger of getting beached? Heather knew some of her students would struggle with the culture of elite swimming, so part of her reading strategy was to provide visual cues, graphic representations, gestures, realia, and pictures to support their understanding, in line with the principles of Universal Design for Learning. She presented the story to the students, wearing a swimming cap, goggles, and sweat suit to class each day, and gave the students data to help them predict the likelihood of the swimmer's survival in different scenarios.

The students were enchanted by the story and spent time synthesizing information from different sources—including scale maps, cold-water survival charts, and an article about swimmer's endurance. Heather's students benefited from her long-term focus on academic vocabulary instruction, which developed confidence in students—especially English learners—to correctly decide which math function they should apply. A focus on vocabulary allowed Heather to address a fundamental aspect of her curriculum; with a language for understanding, students in this activity persevered at organizing data into new formats: number lines, function tables, and coordinate planes. The students mapped the swimmer's different paths that changed in rate due to ocean current. The lessons explored many different mathematical content connections (in exploring changing quantities) and practice areas. The students analyzed proportional relationships, added fractions, used ratio reasoning to solve problems, compared two different functions, and made use of data. They also persevered in solving a complex problem, constructed viable arguments, and critiqued the reasoning of others.

Figure 7X and Figure 7X





Figures 7X and 7X (above) include students' work showing a current moving 3/4 of a knot against the swimmer as she swims back to the pier. The current against her changes her rate of progress to 1 1/4 of a mile per hour. Students used the different rates, which they displayed in tables and graphs.

The week before the whale project Heather created an ocean scene in her classroom—complete with realia in the form of a cutout of a baby whale. The students researched the names and dimensions of the sea animals that would appear in the story, and practiced precision with measurement. The students measured, drew, and cut out the animals to create an ocean scene, but Heather kept the whale story project a surprise.

The approach of investigating mathematical ideas to make sense of the world, to predict and to impact the future should be the goal for mathematics lessons. These investigations include connected content and mathematical practices. In the following section we explain the four content connections, with examples for each.

CC1: Communicating Stories with Data

Grades 6–8 mathematics courses should give prominence to statistical understanding, reasoning with and about data, reflecting the growing importance of data as the source of most mathematical situations that students will encounter in their lives. Drivers of Investigation will allow students to understand and explain, predict, and affect the world using data that is generated by students, or accessed from publicly-available sources. Such data investigations will help maintain and build the integration of mathematics with students' lives (and with other disciplines such as science and social studies). investigations in this category will also draw from other content connections such as: Exploring changing quantities. An example of a data investigation attending to the integration of different subjects and ELD is described in the following Snapshot.

Snapshot:

In this example, students observed flocks of crows and seagulls hovering over the lunch area by the cafeteria around nutrition and lunch times. During a campus-wide survey and mapping activity, students observed large amounts of trash and food waste. They wanted to study the effects of this waste on the health of students and teachers as well as the local and regional natural systems and local community.

Their teacher was focused on having students generate authentic questions and conduct an investigation of the campus community to deepen their knowledge and skills in math, science, English language arts, and align this investigation with California's Environmental Principles and Concepts (EP&Cs). She saw this as an opportunity for students to communicate stories with data by building awareness of the connections between mathematical ideas and environmental and social justice issues on campus and in their local community.

The math-related focus of this investigation had students collect data from the lunch areas and cafeteria as a way to make the assignment local, relevant, and meaningful to their daily lives. The teacher decided to focus on content related to statistics and probability by having students use random sampling to draw inferences about a population (7.SP.1, 7.SP.2) and draw informal comparative inferences about two populations (7.SP.3, 7.SP.4).

From a science perspective, student work was to focus on: planning and carrying out an investigation (CA NGSS SEP-3); analyzing and interpreting data (CA NGSS SEP-4); using mathematical and computational thinking (CA NGSS SEP-5); constructing explanations and designing solutions (CA NGSS SEP-6); examining the cycling of matter and energy transfer in ecosystems (CA NGSS 7.LS2.B), and, developing possible solutions (CA NGSS 7.ETS1.B).

Students would analyze the results of their investigation to examine how “the long-term functioning and health of terrestrial, freshwater, coastal and marine ecosystems are influenced by their relationships with human societies” (CA EP&C II); ”the exchange of matter between natural systems and human societies affects the long‑term functioning of both” (CA EP&C IV); and, how “decisions affecting resources and natural systems are based on a wide range of considerations and decision‑making processes (CA EP&C V).

Based on their investigations, mathematical analysis, and a consideration of the environmental principles, students would “write an informative/explanatory text(s), including the narration of… scientific procedures/experiments, or technical processes” (ELA WHST.6-8.2.a-f), and cite specific textual evidence to support analysis of science and technical texts (ELA RST.6–8.1).

During an initial exploration of campus, students observed large numbers of crows and seagulls hovering over the lunch area by the cafeteria. They noticed that the number of birds was largest just after lunch.

Back in the classroom, the teacher wanted to give students opportunities to generate authentic questions about what they observed in the lunch area. She asked students what they wonder about the situation, and noted their wonderings on the board. Examples include: when are the largest numbers of birds in the lunch area; what is attracting the birds; and, do students at different grades produce different amounts of food waste and trash.

Working in small groups, students generated several questions, ultimately settling on three to reflect the fact that students/grades eat lunch at different times: Do students in different grades produce the same amounts and types of food waste and “trash”? Do students in different grades deal with food waste and “trash” in the same way? Are there different numbers of birds in the lunch area when different grade-level students are eating?

Prior to having students design their investigation and plan how they would collect and analyze data, the teacher introduces the ideas about using random sampling to draw inferences about a population, and how this will allow them to draw informal comparative inferences about the populations of students in the three grades. Using this background information, she guided students in designing a waste audit of trash and food in the lunch area.

After collecting and analyzing their data, the class was able to begin drawing inferences about the amounts and types of food waste and “trash” that students in different grades produced. They determined that students in different grades discarded their food waste and “trash” in different ways. They were also able to determine whether larger numbers of birds visited the lunch area when different grade-level students were eating.

The findings from their investigation resulted in many other wonderings from the students, for example, how the food waste and trash might be: affecting students and people living near the school; the plants and animals on and near the campus; local water quality; the town’s litter prevention program? The teacher suggested that they bring their questions to science class so that they could expand their studies and work together to explore and implement possible solutions.

As part of her strategy for teaching students about the cycling of matter and energy transfer in ecosystems and developing possible solutions, the science teacher had students examine the effects of food waste and trash. She then gave the challenge of using the engineering design process to develop a solution to the problems they identified related to the effects of food waste and trash on students, staff, teachers, the campus, community, and local natural systems.

Noting the students’ enthusiasm about their designs for possible solutions to the food waste and trash problem, the math and science teachers met with the English language arts teacher. They asked him to develop an activity through which students would describe their data collection and statistical analysis, the scientific procedures/experiments they conducted, and the library research that had led them to creating an engineering solution to the lunchtime waste problem.

Each student team was asked to develop both a written description and an oral presentation about their project activities, citing specific textual evidence to support their analysis of the math and science they used to develop their design solutions. As part of the assignment, they were also asked to discuss what they had discovered about the effects of trash and waste on the long-term functioning and health of plants, animals, and natural systems. Students then had the chance to present their work and design solutions to other students, the school administration, and the facilities staff.

Middle school includes a big expansion in important ideas in data science, including:

- Data in the world: exploration, interpretation, decision making, ethics
- Statistical variability: Describing, displaying, and comparing
- Sampling to understand a population: randomness, bias, how many?
- Are they related? Multivariate thinking
- What are the chances? Probability as the basis for data-based claims

As in earlier grades, students experience data science as a tool to help understand their worlds via a process that begins with wondering questions. This is also the beginning of the mathematical modeling cycle (Pelesko, 2015) and the statistical and data science exploration process, and of investigations in science (NGSS Lead States, 2013). (see also Chapter 5).

In sidebar: What is a Model?

Modeling, as used in the CA CCSSM, is primarily about using mathematics to describe the world. In elementary mathematics, a model might be a representation such as a math drawing or a situation equation (operations and algebraic thinking), line plot, picture graph, or bar graph (measurement), or building made of blocks (geometry). In grades 6–7, a model could be a table or plotted line (ratio and proportional reasoning) or box plot, scatter plot, or histogram (statistics and probability). In grade 8, students begin to use functions to model relationships between quantities. In high school, modeling becomes more complex, building on what students have learned in K–8. Representations such as tables or scatter plots are often intermediate steps rather than the models themselves. The same representations and concrete objects used as models of real life situations are used to understand mathematical or statistical concepts. The use of representations and physical objects to understand mathematics is sometimes referred to as "modeling mathematics," and the associated representations and objects are sometimes called "models."

End sidebar.

Source: K–12 Modeling Progression for the Common Core Math Standards, http://ime.math.arizona.edu/progressions/.

The CA CCSSM articulate a range of new standards for data literacy, statistics and data sense-making in the middle grades, some of which are new to teachers—who were likely not taught this content themselves. For this, reason Youcubed developed a set of free data science lessons for students in grades 6–10 (https://www.youcubed.org/data-science-lessons/), a high school course in data science (https://www.youcubed.org/resources/high-school-data-science-course/) and an accompanying online course that is designed to excite teachers about the ideas of data science and teach the main statistical ideas for teachers to be more confident in teaching through data investigations (https://www.youcubed.org/21st-century-teaching-and-learning/). The online course also shares ideas from neuroscience and mindset. The lessons set the content teachers need to know and teach in the broad context of data science as described in Chapter 5, introducing students to investigations in data science and statistics—allowing them to "communicate stories with data."

One important aspect of data literacy teachers can develop in students is an awareness of the ways students are, themselves, surrender personal data, whether through an app, online purchases, or interactive video games. Students should also develop an understanding of the new and creative ways data can be displayed beyond bar graphs or pie charts. Instruction in these lessons can start with a data talk, modeled after a number talk, that begins with a complex data visualization. Ask students: What do you notice? What do you wonder? What is going on in this graph? The New York *Times* section “What is Going on in this Graph?” (https://www.nytimes.com/column/whats-going-on-in-this-graph) provides examples of current, topical, and novel representations of data that serve as strong examples for data talks. Youcubed also offers different data visualizations including many appropriate for elementary and middle school grades (see https://www.youcubed.org/resource/data-talks/).

Data talks provide a space for students to consider and interpret a variety of data and data representations in a low-stakes, exploratory environment. An ideal data visualization is one that offers interest and relevance to students, and also displays data in a way that is new to students or that might have some quirks or features that make the visualization harder to read (and often more engaging!). Show students the visualization, provide any necessary supports and scaffolds and time to process it, and discuss what they notice. These can be observations about how the visual is structured, a question the data is answering, or a lingering curiosity that is raised by the data or that the visual doesn’t address. Teachers do not need to be experts in the content that is displayed; it can be strategic for teachers to field questions from students that they cannot answer if it provides opportunities to model curiosity that comes when an answer is not known. This demystifies the notion that the teacher represents limitless knowledge, and reinforces the ways understanding is ongoing and that curiosity is an opportunity to understand more.

Add Sidebar: Meaning of Relevance

A relevant task invites students to solve problems that are meaningful to them, that relate to their lives or pique their interests. When tasks have cultural relevance they help students to affirm and appreciate their own culture and the cultures of others.

Tasks that help students develop critical consciousness are those that give students the opportunity to understand, critique and solve the problems that result in societal inequalities (Ladson-Billings, 1995 cited in: https://www.cde.ca.gov/pd/ee/culturalrelevantpedagogy.asp).

Figure 7X: Three Components of Culturally Relevant Pedagogy

1. **Student Learning**: The students' intellectual growth and moral development, but also their ability to problem-solve and reason.
2. **Cultural Competence**: Skills that support students to affirm and appreciate their culture of origin while developing fluency in at least one other culture.
3. **Critical Consciousness**: The ability to identify, analyze, and solve real-world problems, especially those that result in societal inequalities.

Chapter 2 links to the 50-state survey of teachers using culturally responsive teaching and the eight competences set out for teachers, also seen in the policy report from Newamerica.org (p. 3): https://www.newamerica.org/education-policy/reports/culturally-responsive-teaching/.

Further detail and ideas for the teaching of data literacy and data science is given in Chapter 5.

Sample Tasks

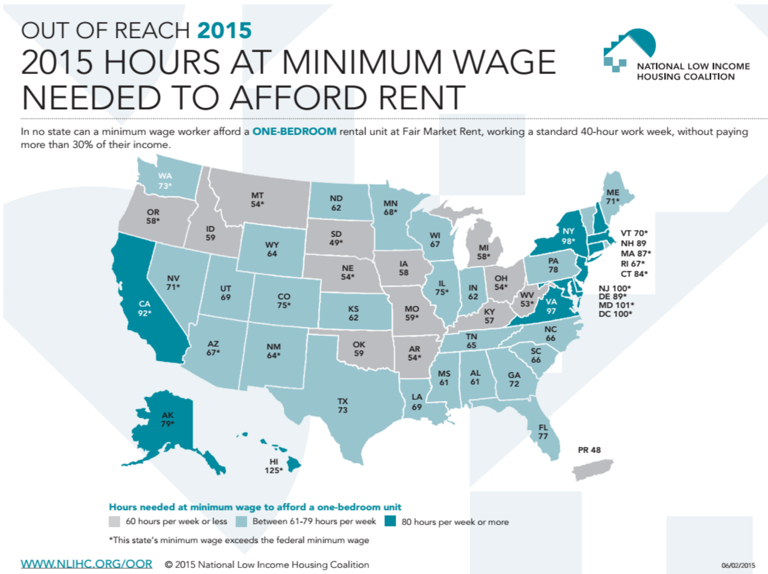
Robert Berry, III, and colleagues have authored a book that brings together lessons that help students "explore, understand and respond to social justice." The lesson "What's a Fair Living Wage?" by Francis Harper, is included in the text. Designed to span 90 minutes, it begins with students discussing what they know about living wages and minimum wages. Students are invited to explore and unpack a data visualization in Figure 7X showing the hours working at minimum wage needed to afford rent in different states in the U.S.

Figure 7X. Hours at Minimum Wage needed to afford rent

- Teachers might initiate the conversation by
  - showing an infographic about living wages/minimum wages:

A **living wage** is a wage that is high enough to maintain a normal standard of living.

A **minimum wage** is the lowest an employer can pay an employee for their work.



Source: © 2015 National Low Income Housing Coalition. Find this year's report at NLIHC.ORG/OOR.

Source: (2015 National Low Income Housing Coalition)

The lesson also includes a video from CNBC.com and a link to a living wage calculator. After students have discussed and consulted different resources teachers can brainstorm a list of questions students have about living wage.

Students then work in groups, with task cards, to consider how many hours each family needs to work to pay rent for the type of apartments best for each family, with focused teacher questions:

Student Task Cards (downloadable from https://resources.corwin.com)

Student Task Cards

RED Family: 1 adult

You are a Filipino American male who just graduated from high school and need to move out on your own. You found a job making minimum wage for nontipped employees in Chicago, $10.50 per hour, as a line cook at a nearby restaurant. You work forty hours per week.

GREEN Family: 1 adult; 1 child

You are a young, single white mom with one child working as a server at a nearby restaurant. Minimum wage is different if you receive tips, $5.95 per hour. You make minimum wage, and you average about $360 per week in tips. You work forty hours per week.

BLUE Family: 2 adults; 2 children

You are a Latinx family with two children under the age of five. Mom stays home to take care of the children. Dad works forty hours per week at a construction company that pays two times minimum wage for nontipped employees.

YELLOW Family: 1 adult

You are a young, single Black woman who is going to school part time and working full time (forty hours per week). You work at the same construction company as the dad of the BLUE family, but most Black women (including you) make 64 percent of what men at the company make.

ORANGE Family:1 adult

You are a Palestinian American female who is a full-time student working about twenty hours per week. You have a minimum wage job working in the library (no tips). However, you also have a scholarship that provides $1, 000 at the beginning of every month.

PURPLE Family: 2 adults; 2 children

You are a two-mom Black family with two children. Both of your children are in school, so both moms work full time (forty hours per week). Both found jobs working for Amazon in Romeoville, Illinois. Amazon pays employees $13.00 per hour.

What's a Fair Living Wage?

Part 1

Today, your group will figure out the hourly wage necessary for a family in Chicago to afford housing. You will look at real data about hourly wages (the amount of money you make per hour) and the cost of renting each month. Your goal is to use mathematics to decide whether or not you think six families in Chicago are paid fair wages.

Your Task: As a team, do the following: Figure out how many hours each family needs to work to pay rent for the type of apartment you think is best for the family.

Guidelines

- Draw a graph and write an equation for each family's earnings over time.
- Use a different color pencil/marker for each family.
- Identify the dependent and independent variables.
- Use the following data about fair housing rental prices for monthly rent:

Studio	1 Bedroom	2 Bedroom	3 Bedroom	4 Bedroom
$860	$1,001	$1,176	$1,494	$1,780

Data source: http://huduser.gov

Your team must work cooperatively to solve the problems. No team member has enough information to solve the problems alone!

- Each member of the team will select a family—Red, Green, Blue, Yellow, or Orange. DO NOT SHOW your card to your team. You may only communicate the information on the card.
- Everyone can see the PURPLE family card.
- Assume there are four weeks in one month.

You might not need to use all the information on your card to solve the task.

STOP

Check in with your teacher before you answer the next questions.

Teachers support students working in groups and ask the following questions:

- What percentage of income do you think people usually spend on housing, food, and other essentials in our area? Why is this fair and just? Financial advisors recommend 30 percent of monthly income on housing.
- According to the National Low-Income Housing Coalition, a family in (YOUR STATE) needs to make $n per hour to afford a moderate two-bedroom home. Based on your experiences and this task, why does this seem reasonable or unreasonable? If not, what hourly wage do you think is necessary (or did you find from the task) for a family to afford a two-bedroom home?
- How did you decide how many hours was enough to pay rent on the graph, the table, and/or the equation? How can you determine how much the [color] family makes if they don't work? How can you determine how much the [color] family makes if they don't work?

- What does it mean when the two colors intersect? Do they make the same wage? Who makes more money? Will other lines cross? How do you know? What would be a fair hourly wage for our city/state/community? How do you know that wage would be fair? Use the graph, table, or equation to explain how you know.

The lesson concludes with students discussing action that can be taken to increase minimum wage, if they find that wages are not fair for their community.

Taking Action

The particular action will depend on what conclusions students make regarding whether or not the wages in their local community, city, or state are fair and livable. Some possible action steps include the following:

- Invite community stakeholders to talk to students about potentially ongoing efforts to increase wages in the community, city, or state. For example, if there are local organizations, such as unions, who advocate for workers, students might reach out to them about ongoing labor justice efforts. Teachers can invite these stakeholders to speak to the class, and students might elect to join ongoing efforts.
- Have students explore various petitions and recent news articles about increasing the minimum wage, then decide whether or not to sign an existing petition or create an informed script, brochure, video, website, social media post, and so on to share what they learned about labor issues with others who might also be interested in signing a petition.
- Have students write their own letters to city, state, or federal representatives, sharing what they learned from this task and advocating for the fair living wage that they determined.
- Have students investigate arguments for and against increasing the minimum wage using the mathematics discovered in the lesson, and hold a mock debate in class. This would allow them to practice communicating with others who might have different views about wage and labor issues, including future employers.

Source: Berry, Robert Q, III; Conway, Basil M., IV; Lawler, Brian R.; Staley, John

W. High School Mathematics Lessons to Explore, Understand, and Respond to Social Injustice (Corwin Mathematics Series) (p. 150). SAGE Publications. Kindle Edition.

Driver of Investigation: Impacting the Future

Content Connections: Exploring Changing Quantities

- CCSS.8.EE.C.8.B

  Solve systems of two linear equations in two variables algebraically, and estimate solutions by graphing the equations. Solve simple cases by inspection. *For example, 3x + 2y = 5 and 3x + 2y = 6 have no solution because 3x + 2y cannot simultaneously be 5 and 6*.

- CCSS.8.EE.C.8.C

  Solve real-world and mathematical problems leading to two linear equations in two variables. *For example, given coordinates for two pairs of points, determine whether the line through the first pair of points intersects the line through the second pair*.

- CCSS.8.F.A.2

  Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change.

- CCSS.8.F.B.4

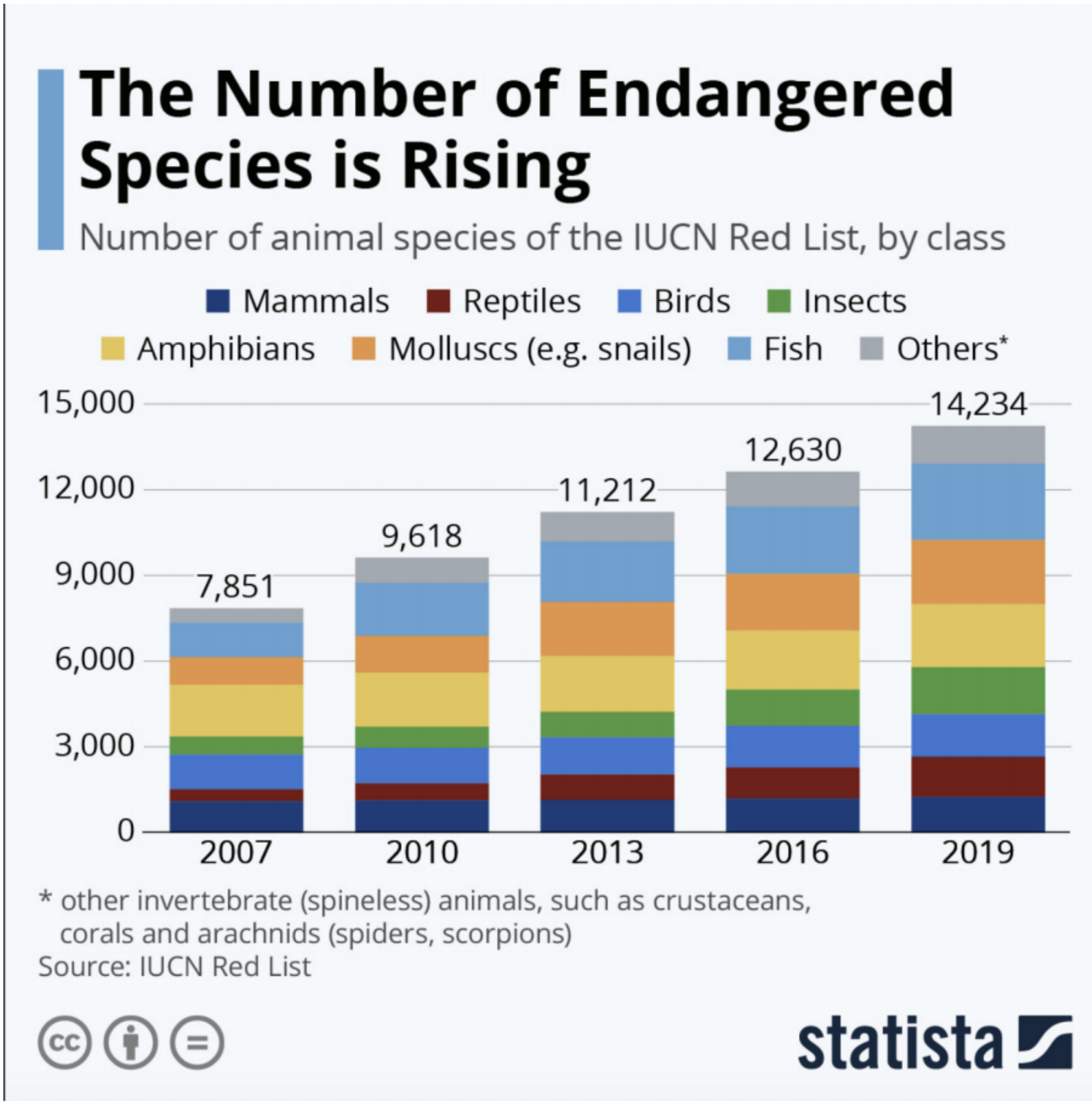
  Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (*x, y*) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.

Mathematical Practices

- CCSS.MP.1 Make sense of problems and persevere in solving them.
- CCSS.MP.2 Reason abstractly and quantitatively.
- CCSS.MP.3 Construct viable arguments and critique the reasoning of others.
- CCSS.MP.4 Model with mathematics.
- CCSS.MP.5 Use appropriate tools strategically.
- CCSS.MP.6 Attend to precision.
- CCSS.MP.7 Look for and make use of structure.
- CCSS.MP.8 Look for and express regularity in repeated reasoning.

Teachers can poll their students to find out interests, and use these interests to both motivate learning and bridge cultural divides in their classrooms. While some students may not readily see the connections between mathematics and sports, for example, data visualization is a powerful means of exploring performances of athletes. An example of a data visualization that students may really enjoy comes from the statistics and analytics website *FiveThirtyEight* showing the basketball shots of basketball player Stephen Curry, as highlighted in Chapter 5. (https://fivethirtyeight.com/features/stephen-curry-is-the-revolution/), another example comes from the National Collegiate Athletic Association (NCAA) sharing Division 1 women soccer games between 2017–2019 (approx. 6500 games) (https://www.youcubed.org/wp-content/uploads/2020/11/Womens-Soccer-1.pdf).

Data visualizations can also highlight important environmental issues such as the following:



https://www.statista.com/chart/17122/number-of-threatened-species-red-list/

Long Description: Bar graphs display with left axis representing number of endangered species, labeled from 0 to 15,000. Bottom axis is in years, 2007 to 2019, with one bar every three years, so 5 bars total. The bars increase in height to indicate that the number of endangered species has risen from 2007 to 2019. The height of the first bar, for year 2007, is 7,851 and the height of the last bar for year 2019, is 14,234. The bars are also color coded to indicate more specificity for 8 types of species: mammals, reptiles, amphibians, birds, insects, mollusks, fish and other. Source: https://www.youcubed.org/wp-content/uploads/2020/09/EndangeredSpecies.pdf

The Youcubed Data Science units, which contain lessons that are written for students from grade 6 and upwards, start with students learning about the meaning of data, and collecting a data diary—recording all the data they give away over a 24-hour period. They are also shown a data science picture book, that invites them to reflect on many aspects of data and data science. In later units students are introduced to a powerful data tool: the Common Online Data Analysis Platform (CODAP) (https://codap.concord.org/) that enables them to explore data sets. Students are encouraged to ask questions of data—which could be a data set teachers import into CODAP or it could be one of the datasets provided. In any of the data investigations students can investigate patterns of association in bivariate data, visually exploring them by dragging two variables to the different axes in the CODAP tool. In unit four, students are invited to collect survey data and compare the data with other previously collected survey data, drawing comparative inferences about two populations.

In these data lessons, students are invited to be data explorers, learning about tools and measures—such as measures of center (mean, mode, median) and spread (standard deviation)—as they investigate questions that they find interesting. For more detail see, visit the Youcubed resource https://www.youcubed.org/data-science-lessons/.

CC1 CA CCSSM Clusters of Emphasis

- 6.SP: Develop understanding of statistical variability. Summarize and describe distributions.
- 7.SP: Use random sampling to draw inferences about a population. Draw informal comparative inferences about two populations. Investigate chance processes and develop, use, and evaluate probability models.
- 8.SP: Investigate patterns of association in bivariate data.
- 8.EE: Understand the connections between proportional relationships, lines and linear equations.*
- 6.RP: Understand ratio concepts and use ratio reasoning to solve problems.

- 7.RP Analyze proportional relationships and use them to solve real-world and mathematical problems.

Mathematical Practices

- CCSS.MP.1 Make sense of problems and persevere in solving them.
- CCSS.MP.2 Reason abstractly and quantitatively.
- CCSS.MP.3 Construct viable arguments and critique the reasoning of others.
- CCSS.MP.4 Model with mathematics.
- CCSS.MP.5 Use appropriate tools strategically.
- CCSS.MP.6 Attend to precision.
- CCSS.MP.7 Look for and make use of structure.
- CCSS.MP.8 Look for and express regularity in repeated reasoning.

CC2: Exploring Changing Quantities

Counting, organizing, adding, subtracting, multiplying and dividing quantities have been of primary importance for much of students' mathematics experiences in TK–5. In grade six, students are introduced to the idea that quantities act in concert, in many cases, rather than alone. Developing an understanding of how quantities can vary together begins with the transition from fractions to ratios. The understanding of fractions established in grades 3–5 provides students with the foundation they need to explore ratios, rates, and percents in grades 6–8. In grade six, students' prior understanding of multiplication and division of whole numbers, and fraction concepts such as equivalence and fraction operations, contribute to their study of ratios, unit rates, and proportional relationships. In grade seven, students deepen their proportional reasoning as they investigate proportional relationships, determine unit rates, and work with two-variable equations. In grade eight, they build on their work with unit rates from grade six and proportional relationships from grade seven to compare graphs, tables, and equations of proportional relationships and form a pivotal understanding for the slope of a line as a type of unit rate. This learning progression culminates in students' introduction to functions in grade eight as one of the most important types of co-varying relationships between two quantities. In a sense, in grades 6–8, students transition from an understanding of quantities as independent to quantities that vary together.

Through investigations in this connected content area, students build many concrete examples of functions. CC2 connects easily with CC1: Communicating stories with data with many rich modeling and statistics investigations. Specific, contextualized examples of functions are crucial precursors to students' work with categories of functions such as linear, exponential, quadratic, polynomial, rational, etc. and to the abstract notion of function. Notice that the name of the COI considers changing quantities, not changing numbers. Functions referring to authentic contexts give students concrete representations that can serve as contexts for reasoning, providing multiple entry paths and reasoning strategies—as well as ample necessity to engage in SMP.2 (Reason abstractly and quantitatively). This embedding also maintains and builds connections between mathematical ideas and students' lives.

Ratios and Proportions

Educational research has focused on students' understanding of ratios and proportional situations for several decades, largely because of the crucial bridge that ratios and proportions form between fractions (in elementary grades) and linear relationships (in high school grades). The type of thinking that children exhibit as they work on proportional situations is known as **proportional reasoning**, which Lamon (2012) defines as "reasoning up and down in situations in which there exists an invariant (constant) relationship between two quantities that are linked and varying together" (p. 3). Lamon (2012) also points out that this type of reasoning goes well beyond simply setting up or solving equations of the form $a/b = c/d$ (see also Chapter 3 for issues that arise when cross-multiplying).

In general, Lamon (1993) characterized two dimensions of proportional reasoning as relative thinking and unitizing. Relative thinking is the ability to compare quantities in problem situations, while unitizing is the ability to shift the perception of the unit (or whole/unit whole) to incorporate composite units. Activities and problems which foster the development of these dimensions should be utilized where possible. In general, emphasis should be placed on students' ability to recognize the connections between representations of the quantities in problems, and the connections between solution strategies, rather than on solely finding answers.

Carpenter et al. (1999) proposed four stages of proportional reasoning development in students:

- Level 1: Students focus on random calculations or additive differences in ratio work.
- Level 2: Students perceive a ratio as a single unit, and can scale up or down the ratio, in a multiplicative or additive fashion, by scale factors that are whole numbers.
- Level 3: Students still conceive of a ratio as a single unit, but they can scale the ratio by nonintegers.
- Level 4: Students recognize and make use of the relationship within a ratio and between two equivalent ratios.

In investigating middle-grades girls' learning of proportions, Steinthorsdottir and Sriraman (2009) provided evidence in support of providing sequenced tasks in consideration of the above levels. Specifically, tasks which facilitated students to think between ratios and within ratios were found beneficial. The norms of productive discourse and provision of appropriate scaffolding further supported positive learning.

Relative Thinking

The approaches to the following task illustrate the relative thinking described by Lamon (2012), and demonstrate a progression from ratio understanding to proportional reasoning by focusing on connections between differing viewpoints of the problem.

Example: Mixing paint, Grade 6

A recipe for Orange Sunglow paint calls for three parts of yellow paint to four parts of red paint. How many cups of yellow are needed to make a batch that uses 20 parts of red paint?

Approach 1: Tape diagrams

A tape diagram (a drawing that looks like a segment of tape) can be used to illustrate a ratio. Tape diagrams are best used when the quantities in a ratio have the same units. For the Sunglow paint problem above, a tape diagram representation is below.

Note that tape diagrams create a powerful visual cue for students to recognize the part to part ratio 3:4 as well as the visualization of both part to total ratios, 3:7 and 4:7. A subtlety of this type of problem is that the units in this case, parts, is a general term. A "part" is a generic label—what is essential is the relative ratio of a number of parts to another number of parts. However, the use of the same general unit, "part," indicates that the *size* of the part must be the same for both types of paint. Distinguishing these intricacies can support understanding for all students, but it is especially necessary that teachers provide English learners with opportunities to understand that vocabulary can have multiple meanings.

One key advantage of tape diagrams is that they can easily be modeled with physical materials that students can manipulate and annotate themselves. Tape diagrams can serve as concrete models, representing specific problems, supporting students as they create abstract or mental representations of these models with additional experiences.

Approach 2: Ratio Tables and Unit Rates

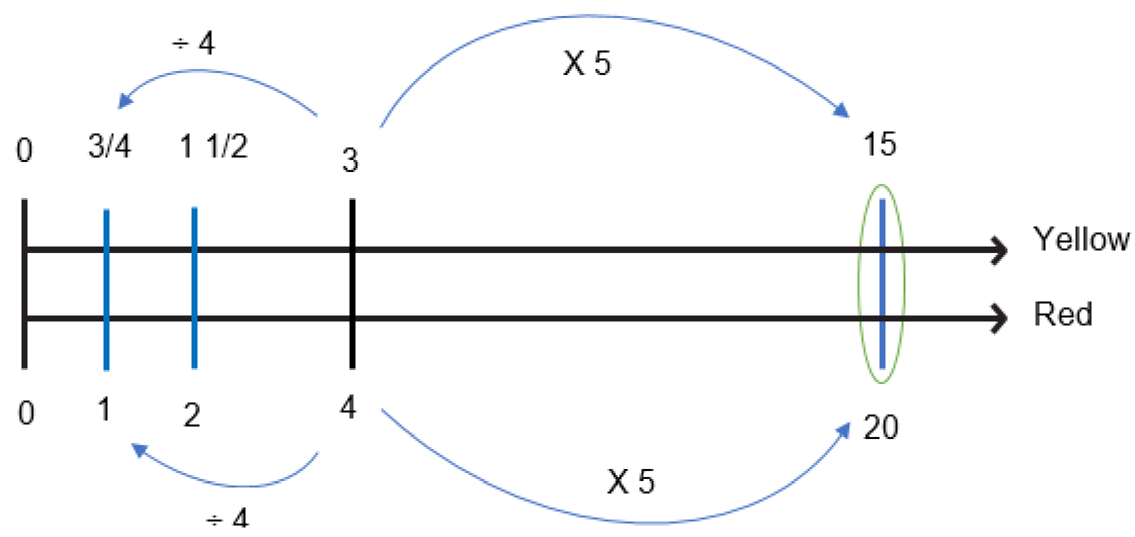
Yellow Parts	Red Parts	Orange Sunglow Parts
3	4	7
[blank]	[blank]	[blank]
[blank]	[blank]	[blank]
[blank]	[blank]	[blank]

Yellow Parts	Red Parts	Orange Sunglow Parts
[blank]	[blank]	[blank]

Ratio tables present equivalent ratios in a table format, and students can use tables to practice using ratio and rate language to deepen their understanding of what a ratio describes. As students generate equivalent ratios and record ratios in tables, they should notice the role of multiplication and division in how entries are related to each other. Students also understand that equivalent ratios have the same unit rate. Tables that are arranged vertically may help students to see the multiplicative relationship between equivalent ratios and help them avoid confusing ratios with fractions. (adapted from Common Core Standards Writing Team, 2019).

The teacher can provide the table above as a starting point and encourage students to discuss and then choose ways to fill in blanks (6.RP.3a). In realizing that equivalent ratios are present in each row, and identifying several pairs of ratios in the table as part to part or part to whole relationships, students' use of ratio language in describing the relationships among entries in the table is strengthened (6.RP.1). Since equivalent ratios express the same unit rate, by dividing entries in any row, unit rates can be found. With a bit of guidance, students can often discover this fact for themselves, as well as the fact that any row in which a one appears exhibits unit rate relationships (6.RP.2). For example, if one Red part were listed, then the rest of the row is 3/4 Yellow parts and 7/4 Sunglow parts. Thus, one Red to 3/4 Yellow is not only in an equivalent ratio, but we could say that there are 3/4 Yellow parts per every one Red part. Similarly, students can recognize that there are 4/3 Red parts for every one Yellow part.

Approach 3: Double Number Lines



A double number line diagram sets up two number lines with zeroes connected. The same tick marks are used on each line, but the number lines have different units, which is central to how double number lines exhibit a ratio. The paint example is represented below, with some of the arrows indicating how to find the appropriate number of yellow parts for 20 red parts, and how the unit rate is calculated. For another more detailed classroom example focused on double number lines, see the grade six vignette in Chapter 3.

Approach 4: Between and Within Ratio Relationships (Extending to 7th and 8th Grade)

In recognizing that scaling up from 4 red to 20 red requires a factor of 5, and then multiplying 3 yellow by the factor of 5, students are employing a **between ratio relationship**. This is sometimes referred to as thinking across the equals sign in the proportional set-up of this problem: 3/4 = y/20.

Students utilizing a **within ratio relationship** recognize that the internal factor of 4/3 characterizes the yellow to red relationship (4/3 the number of yellow gives the number of red). From the reverse direction, red to yellow, the within ratio relationship recognizes that the internal factor is 3/4 (3/4 the number of red gives the number of yellow). Employing this second within ratio relationship would enable a student to determine 20 red times 3/4 must result in 15 yellow.

Not only are 4/3 and 3/4 also the unit rates (as described in Approach 2 above), in seventh grade, students recognize these numbers, 4/3 and 3/4 as the **constants of proportionality**. In eighth grade, as students understand these values as conversion factors between red and yellow, they can create equations R = 4/3*Y and Y = 3/4*R. Moreover, as students look to graph these relationships in the coordinate plane, they can utilize these unit rates/constants of proportionality/conversion factors as the measures of the steepness of lines in the coordinate plane, since the slope of each line is precisely the ratio of red to yellow or yellow to red. Thus, a strong understanding of ratio relationships provides the basis for understanding slope, one of the most crucial ratios for students to understand in high school.

Illustrative Mathematics show a progression of representations from sixth to eighth grade moving from drawings and double number line diagrams in sixth grade (https://im.kendallhunt.com/MS/families/1/2/index.html), to tables in seventh grade (https://im.kendallhunt.com/MS/families/2/2/index.html) and bivariate graphs in eighth grade (https://im.kendallhunt.com/MS/families/3/3/index.html).

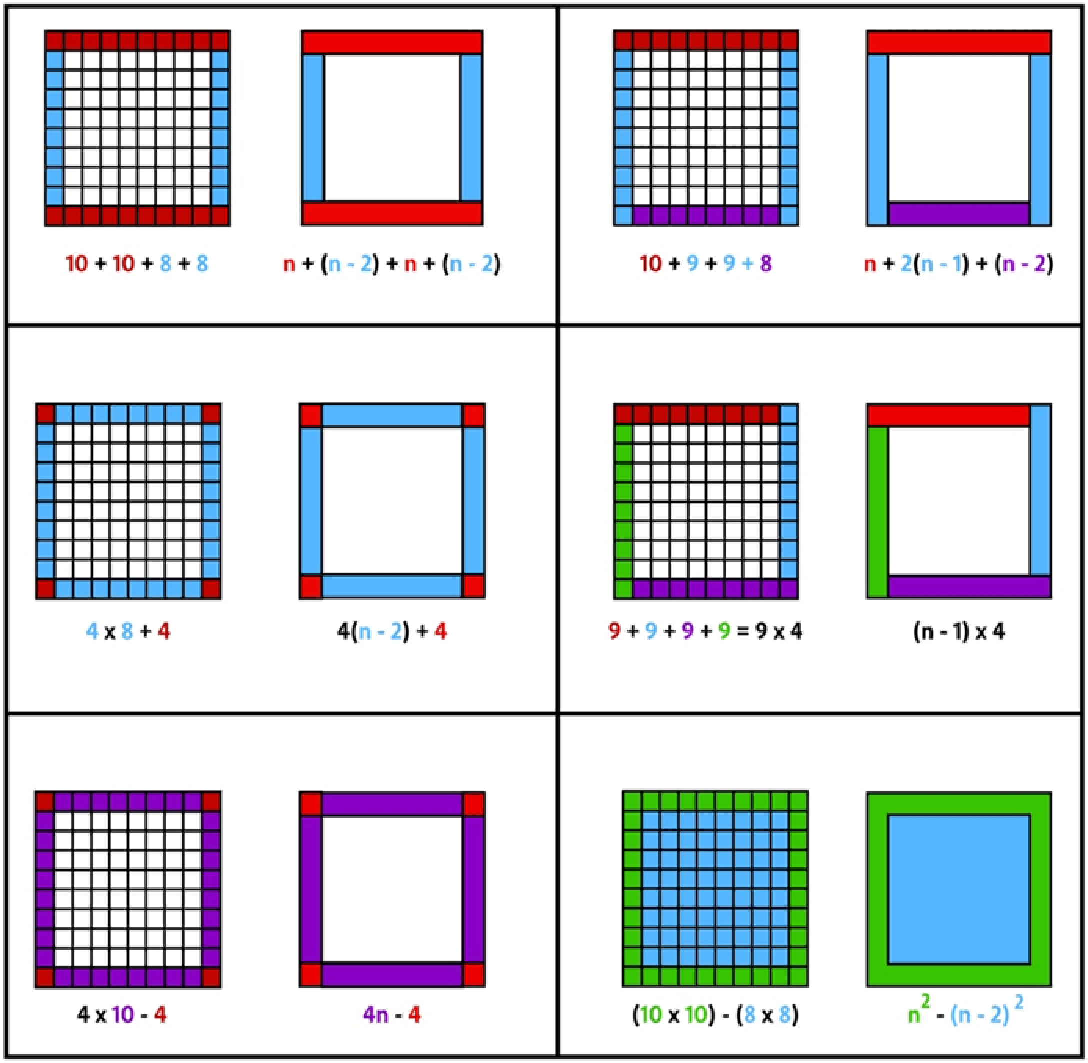
Note that since steepness is such a commonly experienced phenomena for children, the use of physical ramps and ramp scenarios can foster a more tactile understanding of ratios and the related concepts of slope, steepness, similarity and proportionality.

Task: The Border Problem—Grades 6–8

Algebra is often taught through symbols and symbol manipulation, but research from neuroscience shows that students benefit from approaching content in different ways. Mathematics that students engage with visually and through words, is especially important to combine with number and symbol work, as it causes important brain connections to develop (Boaler et al, 2016). Algebra that is approached visually also enables students to see mathematics as a creative and connected subject. One of the most well-known and effective lessons for introducing students to algebra visually, and for helping them understand algebraic equivalence is the border problem.

In this activity, students are asked to look briefly at a border around a square and work out how many squares are in the border, without counting them (see also Boaler & Humphreys, 2005). It is important to show the border only briefly to prevent students from having time to count the squares. Students determine many different answers for the number of squares on the border—40, 38, and the correct answer of 36 are typical. A variety of responses—along with the teacher's specific open-ended questions and academic conversation sentence frames—allows teachers the opportunity to ask students to justify different answers, to construct viable arguments and critique the reasoning of others. As the lesson progresses, students think numerically and then verbally and eventually algebraically about ways to describe the number of squares in any border and the different ways in which students see the number. These different ways of seeing the border offers an opportunity for seeing and understanding algebraic equivalence.

Figure 7X



Source, which includes a full lesson plan for the border problem is available online at https://www.youcubed.org/wp-content/uploads/2018/09/Border-Problem-final-copy.pdf.

Vignette: Equivalent Expressions—Integrated ELD and Mathematics

Background: Mr. Garcia's sixth-grade class recently started a unit on expressions and equations. The class has explored the difference between equations and expressions. They have also been using the properties of operations to generate equivalent expressions and determine if two expressions are equivalent. Mr. Garcia's class of 32 students includes four students with an Individualized Education Plan (IEP) and eight students who are English learners. Of these students, one is at the Bridging level, five are at the Expanding level, and two are at the Emerging level. Sal, one of his students at the Emerging level, is a newcomer who joined the class several weeks ago after moving to the United States from Mexico. Each of the four sixth-grade classes are similar in their composition of English learners, with between eight and 10 per class.

Mr. Garcia meets weekly with the other three self-contained sixth-grade teachers to collaborate. During this time, they discuss relevant student data, upcoming units of instruction, and areas of focus for designated and integrated English Language Development (ELD) instruction when they deploy their students to receive specialized instruction (see the additional designated ELD resources below). They also discuss the strengths of their students who are acquiring English, or who have IEPs, and plan the ways they will build upon their strengths. The teachers know that diversity enriches all student conversations, especially when students are given multiple different ways to access ideas—through visuals, physical manipulatives, and supportive discussions. The teachers use of multiple forms of engagement, representation, action and expression in their mathematics teaching is aligned to the Universal Design for Learning (UDL) guidelines: http://udlguidelines.cast.org/?utm_source=castsite&lutm_medium=web&utm_campaign=none&utm_content=aboutudl. The discussions the teachers plan give language learners and all students opportunities to access the language of mathematics in a supportive environment, learning mathematical ideas and mathematical language together (Zwiers, 2018).

Lesson Context: The sixth graders are several lessons into their unit on expressions and equations. Mr. Garcia has been working with his students to create equivalent expressions and to determine whether or not two expressions are equivalent. He wants to use a particular lesson to employ formative assessment strategies that allow him to gauge his students current level of understanding with this concept and determine areas of need to guide his next steps. To do this, he has selected an *Illustrative Mathematics* task where students will have to determine which student expressions are equivalent and justify their thinking. He hopes that this lesson will serve to deepen student understanding about equivalent expressions by connecting them to a familiar context, the perimeter of a rectangle. He also believes that this context will be useful for guiding conversations about why expressions are equivalent based on the structure of the rectangle and the parts of the expressions. Mr. Garcia plans to ask students to justify the equivalence of the expressions by connecting the expression to the labeled picture of the rectangle.

Lesson Excerpts: Mr. Garcia's lesson engages students in analyzing given expressions to determine if they are equivalent. The task also includes a context with a visual support to encourage students to connect the expressions to the corresponding elements in the visual representation. Mr. Garcia knows that the multi-model forms of mathematical expression will support the learning of students with learning differences as well as those who are English learners—and other students. He is curious about whether or not students understand that different equivalent expressions can illustrate different aspects of the same situation. He wants to determine which students have internalized the academic language and use it naturally to explain their thinking.

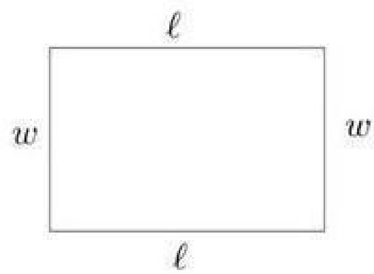
Learning Target: The students will analyze different student expressions for the perimeter of a rectangle to determine if the expressions are equivalent and they will justify the equivalence in conversations and in writing.

CA CCSS for Mathematics: 6.EE.4 - Identify when two expressions are equivalent (i.e., when the two expressions name the same number regardless of which value is substituted into them). For ex., the expressions y + y + y and 3y are equivalent because they name the same number regardless of which number y stands for; SMP 7 - Look for and make use of structure; SMP 3 - Construct viable arguments and critique the reasoning of others.

CA ELD Standards: ELD.PI.6.1 - Exchanging information and ideas with others through oral collaborative discussions on a range of social and academic topics; ELD.PI.6.11 - Justifying own arguments and evaluating others' arguments in writing.

Mr. Garcia planned the lesson to encourage many opportunities for students to learn the language of mathematics and support the development of English proficiency through a variety of academic conversations in new contexts, paired with the support of visual representations.

Mr. Garcia begins the lesson by showing students the image below and asking them to write an expression for the perimeter of this rectangle using the given variables. He begins in this way in order to connect to what students have learned about creating expressions since the beginning of the unit. He believes that having students create their own expressions first will allow them to create a foundation for forming their arguments about whether or not the other expressions in the task represent the perimeter of the rectangle.



After students have created an expression for the image, Mr. Garcia asks them to share their expressions with their table groups. Mr. Garcia has made groups using his knowledge of the different students in his class, grouping students together who can support each other's learning. He does not place students in groups according to the support they may require—language learning or learning differences—but focuses on creating groups where varied and different strengths complement one another. He asks the groups to briefly discuss whether their expressions are the same or different, and if they are different, if the group believes that they are equivalent or not.

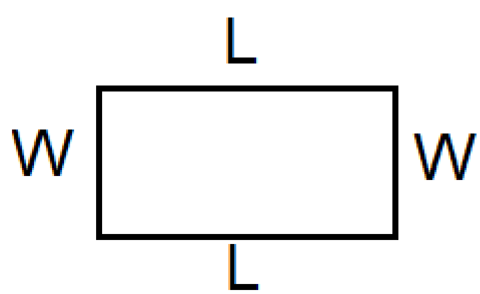
Mr. Garcia then conducts a "collect and display" by scribing student responses (using their exact words and attributing authorship) on a graphic organizer on the board. He asks specific questions about the different-looking representations such as, "Where is the 2w in this picture?" "Which term represents this line on the rectangle?"

Mr. Garcia: I want you to think about the expression you wrote and the other expressions that were shared at your table. Using what you have learned about equivalent expressions, expressions that mean the same thing and have the same value, I want you to explore this task.

Next, Mr. Garcia provides students with a sheet with expressions for the same task that have been proposed by students in another class (see Figure X). He reads the task aloud as students read along on their own copies of the task. As Mr. Garcia reads, students mark the text to indicate important information, ideas, and questions they may have.

The students in Mr. Nolan's class are writing expressions for the perimeter of a rectangle of side length L and width W. After they share their answers, the following expressions are on the board.

- Sam: 2 (L + W)
- Joanna: L + W + L + W
- Kiyo: 2L + W
- Erica: 2W + 2L



Which of the expressions are correct and how might the students have been thinking about finding the perimeter of the rectangle?

After posing the task, Mr. Garcia provides the students several minutes of independent time to think about and work on the task to determine which expressions correctly represent the situation and why. Students are given several minutes to work on the task independently. Next, Mr. Garcia asks the groups to discuss which of the expressions are correct and justify their thinking. He circulates around the room while groups are discussing their ideas, making notes about what he is hearing to inform his formative assessment process and helping him determine which students he might be willing to share.

Mr. Garcia: As I walked around the classroom, I heard students using the word equation and expression interchangeably to mean the same thing. Before we share ideas about the task, I want your groups to discuss whether or not equation and expression mean the same thing, and if not, how are they different?

Mr. Garcia stops at one of the tables to listen to their discussion. He tells the table group that he would like them to share their conversation with the class and he asks Cecily, an English learner at the Expanding level, if she would be willing to share for the group. She agrees and he asks her to practice what she will say with her group before sharing with the whole class.

Mr. Garcia: As I listened to table groups, I heard conversations explaining the difference between expressions and equations. I have asked Cecily to share Table 4’s ideas with the class.

Cecily: My group discussed how equations and expressions are different. We think that equations have equal signs and expressions do not.

Mr. Garcia: Can anyone add on to what Cecily said? Alex.

Alex: My group agreed with Cecily’s group and we also said that an equation shows two expressions that are equal to each other. The expression on one side equals the expression on the other side.

Mr. Garcia: Okay, so Alex, you’re saying that if 5x is an expression (Mr. Garcia writes this on the whiteboard and labels it expression) then 5x = 4x + 2 is an equation (Mr. Garcia writes this on the whiteboard and labels it equation), correct?

Alex: Yes, an equation is made up of two expressions.

Mr. Garcia: Now that you’ve heard some ideas about the difference between expressions and equations, please tell your partner what you have learned.

Students discuss the difference between expressions and equations with their partner as Mr. Garcia again walks around the classroom to gauge understanding in partner discussions. He intentionally visits two partner groups where one of the partners is an English learner to see if these students are understanding the conceptual difference behind these two math terms. Through structured language support that utilizes the key math terms of the lesson, students can construct verbal and written responses to show their learning.

Next, Mr. Garcia brings the class back together to have a class conversation about the task. He asks students to share a correct expression and explain how the parts of the expression relate to the picture. Mr. Garcia has also been using talk moves with his class to strengthen their classroom discussions and makes a conscious effort to model and use these moves throughout the discussion. Recently, he has been focusing on supporting the talk moves of reasoning and turn and talk.

Mr. Garcia: Looking at today's task, can you share an expression that is correct and explain why you believe that it's correct? (Mr. Garcia, give the students some time to think and refer to their work.) Okay, who would like to share? Gabby.

Gabby: (Referring to her work.) I think that Erica is correct because $2w + 2l$ means that there are 2 widths and 2 lengths.

Mr. Garcia: When you say that there are 2 widths and 2 lengths, can you show us what you mean using this picture of the rectangle? (Mr. Garcia points to where the task is displayed by the projector.)

Gabby: Sure. (Gabby walks to the front of the room and points.) The two widths are the sides on the left and right. The two lengths are the top and the bottom.

Eduardo: Well, then why doesn't the equation say $w + w + l + l$?

Mr. Garcia: Class, is there an expression that has it written the way Eduardo suggested? (Note: When Mr. Garcia asks his question, he correctly uses the term expression instead of equation as Eduardo did. He decides to make this gentle correction by restating with the correct term and makes a note to listen to Eduardo's partner conversation to see if he truly understands the concept and term expression.)

Gabby: Yes, Joanna's way shows it like that. It's just in a different order.

Mr. Garcia: So, if Joanna's way, her expression, shows what Eduardo mentioned, turn and talk to your partner about which property you could use to rewrite $l + w + l + w$ *as* $w + w + l + l$ and how you know this property would work?

Students discuss the property they would use to demonstrate that $l + w + l + w$ and $w + w + l + l$ are equivalent expressions. As they are discussing, Mr. Garcia walks to Eduardo's group to listen to how Eduardo explains his thinking. He hears Eduardo use the term expression correctly in his explanation and makes a note to continue to reinforce this concept with students during the duration of the unit as he notices that some students are continuing to struggle accurately to use these math terms.

Mr. Garcia has pre-selected two groups to share their ideas about which property can be used to rewrite the expression. One of these groups includes a student that has struggled recently, so Mr. Garcia wants him to be able to share his ideas with the class to demonstrate his success with this idea. He also asks a pair of girls to share that have not shared a math idea with the class during the last several lessons. Mr. Garcia wants to create opportunities where all student voices are heard and valued, so he carefully selects and records which students share their ideas during math class. As the two pairs share with the class, he asks each group to justify their reasoning by explaining how they know that the commutative property allows them to change the order of an addition expression.

Mr. Garcia: Now that we've talked about two of the equivalent expressions, I'd like to see if there are any expressions from the list that are not equivalent.

Jordan: I think that Kiyo's expression is wrong.

Mr. Garcia: OK, Jordan, since Kiyo isn't here to explain her thinking, can you explain what Kiyo might have been thinking to come up with the expression $2l + w$?

Jordan: I think Kiyo included the top and the bottom, but just didn't go all the way around.

Mr. Garcia: Thank you, Jordan. Who agrees with Jordan that Kiyo's expression is incorrect? (Students show their agree or disagree silent signal.) I see that the majority of the class agrees with Jordan. Please turn and talk with your partner about why you agree or disagree with Jordan. (Mr. Garcia provides time for students to talk with their partners.) Is there anyone who would like to share why you agree or disagree? Sara?

Sara: Well, we agree with Jordan because we just tried a rectangle that is 7 inches long by 4 inches high, and Kiyo's expression says 18 but it's really 22.

Mr. Garcia: Oh, so you tried a specific example. Who else tried an example? (Several hands go up.) That's an important strategy to keep in mind. Emilia, I heard you talking about a different idea with your partner. Do you agree with Jordan?

Emilia: I agree with Jordan that Kiyo is incorrect because she has $2l$, but she only has $1w$, so I think that she forgot one of the widths.

Mr. Garcia: Can you show us what you mean on the screen?

Emilia: Sure. These are her two lengths and she only wrote w, so she has 1 width included, but she forgot this one (pointing to the other side).

Mr. Garcia: Please repeat what Emilia said to your partner. (Students turn and talk to repeat Emilia's idea.)

After students have repeated Emilia's idea, Mr. Garcia shares several ideas and key points that he has heard from students during the lesson. He refers to the examples on the board from earlier in the lesson illustrating the difference between an expression and an equation. He also elaborates on several of the student ideas to connect to the mathematical goal of today's lesson. Next, he draws the class' attention to two sentence frames that he has written on the board and tells students that they may choose to use these frames or they can create their own sentences to begin their writing today.

Sentence Frames:

- [blank] and [blank] are equivalent expressions because [blank].
- The expressions [blank] and [blank] are equivalent because [blank].

Mr. Garcia: On the back of your task, I would like you to select two of the expressions that are equivalent and explain how you know they are equivalent. Please include numbers, words, and pictures to strengthen your explanation.

Students know that the expectation is to write several sentences as needed to completely explain their thinking and that these frames serve as an optional starting point for their writing. Mr. Garcia provides several minutes for students to complete their writing. They also know that in mathematics, their writing is supported through the use of expressions and/or visuals. He wraps up class by having students read their writing to their partner, provide feedback, and revise their writing as needed. Students turn in their writing to end the class session.

Next Steps: Mr. Garcia reads through the student explanations and sorts them into two piles: Got It and Not Yet (Van de Walle, 2005). He looks at the responses in the Not Yet pile to understand students' mathematical thinking to inform his next instructional moves. He discovers that a group of his students are having difficulty justifying equivalence through use of the distributive property, making errors while distributing. He decides to support this small group of students by working with them at the back table over the next several days.

Mr. Garcia also decides to recheck the Got It pile and finds that students were less likely to choose to explain the equivalence of expressions using the distributive property, making him think that this may be an area for growth for the class overall. Based on this, he decides (instead of just working with the Not Yet students) that the whole class would benefit from further work on the distributive property. The structure he chooses is a “re-engaging lesson” (Inside Mathematics, n.d.). This lesson structure uses student work for the purpose of uncovering incomplete understanding, providing feedback on student thinking, helping students go deeper into the mathematics, and encouraging students to reflect on their own learning. Re-engaging is an alternative to reteaching, in which the teacher simply selects a different activity to try to get at the mathematical target of the lesson.

There are several possible activities that fit within this re-engaging lesson structure (San Francisco Unified School District Mathematics Department, 2015). These include brief math talks; a Math Hospital in which the teacher compiles common mistakes and students in teams then identify the errors, diagnose why the errors are common, and correct the errors; and highly-structured Formative Re-engagement Lessons as designed by the Silicon Valley Mathematics Initiative (Inside Mathematics, n.d.).

In this case, Mr. Garcia chooses to hold a math talk using visual models to reinforce the distributive property, then a Math Hospital based on their own work. He is pleased to see that many of his students recognize their own errors represented on the “common errors” sheet, and have good conversations about the sources of mistakes and possible fixes.

As Mr. Garcia continues to teach the lessons in the expressions and equations unit, he uses what he learned about his students from this lesson to connect ideas and deepen student understanding of equivalent expressions. In this extract Mr. Garcia gives the students opportunities to write expressions, compare and contrast those expressions, compare and contrast the ideas of equation and expression, relate expressions to pictures, explain why they agree or disagree with a claim, justify their reasoning about each of them, and examine and correct common errors that arose. These are all rich opportunities for students to use language and for Mr. Garcia to learn about their thinking and language use.

Source: Task: “Rectangle Perimeter 2,” *Illustrative Mathematics*, Cluster 6.EE.A Apply and extend previous understandings of arithmetic to algebraic expressions. https://www.illustrativemathematics.org/content-standards/6/EE/A/4/tasks/461.

Resources

“Expression vs. Equation,” Ask Dr. Math, Math Forum at Drexel, https://www.nctm.org/tmf/library/drmath/sets/mid_equations-1.html?s_keyid=48409881&f_keyid=48409882&start_at=41&num_to_see=40

Chapin, Suzanne H., O’Connor, Catherine, & Canavan Anderson, Nancy. (2013). *Classroom Discussions in Math: A Teacher’s Guide for using talk moves to support the Common Core and more, Third Edition.* Sausalito, California: Math Solutions.

Kazemi, Elham & Hintz, Allison. (2014). *Intentional Talk: How to Structure and LeadProductive Mathematical Discussions.* Portland, Maine: Stenhouse Publishers.

Smith, Margaret S., & Stein, Mary Kay. (2011). *5 Practices for Orchestrating Productive Mathematics Discussions.* Reston, Virginia: The National Council of Teachers of Mathematics, Inc.

Van de Walle, John A., and Sandra Folk. *Elementary and Middle School Mathematics: Teaching Developmentally.* Toronto: Pearson Education Canada, 2005.

William, Dylan. (2011). *Embedded Formative Assessment.* Bloomington, Indiana: Solution Tree Press.

Companion Documents

Equivalent Expressions Designated ELD Connected to Mathematics in Grade Six
Equivalent Expressions Designated ELD: Math & ELD 5-Day Lesson Plan D-ELD 6th

Additional Information

This Integrated ELD and Mathematics Instruction Vignette was adapted from one created by the Tulare County Office of Education under the Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License, http://creativecommons.org/licenses/by-nc-sa/4.0/deed.en_US.

CC3: Taking wholes apart, putting parts together

Students enter middle school with many experiences of taking wholes apart and putting parts together:

- Decomposing numbers by place value
- Assembling sub-products in an area representation of two-digit by two-digit multiplication
- Finding area of a plane figure by decomposing into rectangular or triangular pieces
- Exploring polygons and polyhedra in terms of faces, edges, vertices, and angles

Decomposing challenges and ideas into manageable pieces, and assembling understanding of smaller parts into understanding of a larger whole, are fundamental aspects of using mathematics. Often these processes are closely tied with SMP.7 (Look for and make use of structure). This Content Connection (CC) spans and connects many typically-separate content clusters in number, algebra and shape and space. Decomposing an area computation into parts can lead to an algebraic formulation as a quadratic expression, in which the terms in the expression have actual geometric meaning for students.

It is common to hear teacher stories of students who “know how to do all the parts, but they can’t put them together.” Mathematics textbooks often handle this challenge by doing the intellectual work of assembly *for* the students (perhaps assuming that by reading repeated examples, students will eventually be able to replicate). Word problems in which exactly the mathematically relevant information is included, sub-problems that lay out intermediate calculations and all reasoning, and references to almost-identical worked examples, are all ways of avoiding—rather than developing—the ability to assemble understanding.

Situations that are presented with insufficient or (mathematically) extraneous information, investigations requiring students to decide how to split up the workload (and thus needing to assemble understanding at the conclusion), and problems requiring piecing together factors affecting behavior (such as the function assembly problems in the high school section of Chapter 4) are all ways to engage in this CC.

This CC can serve as a vehicle for student exploration of larger-scale problems and projects, many of which will intersect with other CCs as well. Investigations in this CC will require students to decompose challenges into manageable pieces, and assemble understanding of smaller parts into understanding of a larger whole. When an investigation is included in this CC, it is crucial that decomposing and assembly is a *student* task, not one that is taken on by teacher or text. One important point-of-view change is that enabled by solving a simpler problem, which focuses the view on essential aspects of a problem that are preserved when simplifying other aspects. Mathematicians also regularly draw visual representations of relationships even when the ideas being explored are not geometric (Su, 2020).

In grades 6–8, this CC will especially support students as they develop understanding of the number system, Pythagorean theorem, scientific notation, and angles.

Unitizing

In this next example problem, the task is for students to unpack the notion of what constitutes the whole (also called the unit or unit whole). While identifying the whole is fundamental to understanding fractions in grades 3–5, as described in Chapters 3 and 6, it also is essential in order for students to make sense of proportional situations (Lamon, 2012).

Snapshot: Building Apartments, Sixth Grade

Background: Ms. K often begins the day with her homeroom students by exploring school and community events and happenings. Today, she notices an article in the local newspaper about how bird nesting houses are being built in the park by the river.

The bird species are highly social and prefer a variety of enclosures to mate and rear their young. Since her class has been working on ratio and proportion problems, she sees an opportunity to connect an understanding of ornithology with an understanding of the relevant mathematics for that week. She asks the class to work with a partner and poses the following situation to her class:

> After analyzing local bird populations of a particular species, scientists determined that, in order to meet the bird community's needs, multi-chamber houses are needed. Every time they build three single-chamber houses, they should build four two-chamber houses and one three-chamber house.
>
> (adapted from Lamon, 1993)

She then asks each pair to draft three questions that might indicate wonder about for the given situation; she collects the questions on the board. She notices many of them are interested in how many total chambers there could be, or in how many total birds can be accommodated. Because she has encouraged them to ask questions, her students are able to develop their natural curiosity about ways that numbers, and groups of numbers, fit together. Three questions in particular seem fruitful to explore, and the class helps her clarify these three questions further. So, she has each pair choose one of the following three questions to investigate further and report back to the class via a small poster in 20 minutes.

1. How many houses of each kind are needed to accommodate a certain number of (like 50, 100 or 150) birds? Is there a pattern between the number of houses and number of birds?
2. How many birds could be accommodated if a certain number (like 50 or 100) of the houses are built? Is there a relationship between the number of birds and number of houses?
3. If the park only allows for a certain number, like 50,100, 150 houses to be built in total, how many of each kind would there be? Is there a relationship between number of houses and how many of each kind?

As students work in pairs, she notices that many of them are drawing tables and diagrams to organize their work. In thinking about this problem, students need to be mindful of the many types of units (groups) involved here: groups of each size house, groups of eight houses, total group of birds, total group of chambers, total group of houses. In attending to these different types of units (or wholes), students develop the understanding that there is flexibility in allocating how many parts are in a whole, and that this flexibility offers a new perspective when engaging in proportional reasoning.

CC3 CA CCSSM Clusters of Emphasis

- 6.NS: Apply and extend previous understandings of multiplication and division to divide fractions by fractions. Compute fluently with multi-digit numbers and find common factors and multiples. Apply and extend previous understandings of numbers to the system of rational numbers.
- 6.EE: Apply and extend previous understandings of arithmetic to algebraic expressions. Reason about and solve one-variable equations and inequalities.
- 7.EE: Use properties of operations to generate equivalent expressions. Solve real-life and mathematical problems using numerical and algebraic expressions and equations.
- 6.RP: Understand ratio concepts and use ratio reasoning to solve problems.
- 7.RP Analyze proportional relationships and use them to solve real-world and mathematical problems.
- 7.NS: Apply and extend previous understandings of operations with fractions to add, subtract, multiply and divide rational numbers.
- 8.NS: Know that there are numbers that are not rational, and approximate them by rational numbers.
- 8.EE: Work with radicals and integer exponents. Understand the connections between proportional relationships, lines and linear equations. Analyze and solve linear equations and pairs of simultaneous linear equations.

CC4: Discovering Shape and Space

Developing mathematical tools to explore and understand the physical world should continue to motivate explorations in shape and space. As in other areas, maintaining connection to concrete situations and authentic questions is crucial and this content area could be investigated in any of the ways—to understand, predict or affect.

Geometric situations and questions encourage different modes of thought than do numerical, algebraic, and computational work. It is important to realize that “visual thinking” or “geometric reasoning” is as legitimate as algebraic or computational thinking; and geometric thinking can provide access more readily to rich mathematical work for some students (Driscoll et al., 2007). The CA CCSSM support this visual thinking by defining congruence and similarity in terms of dilations and rigid motions of the plane, and through its emphasis on physical models, transparencies, and geometry software.

As emphasized throughout this framework, flexibility in moving between different representations and points of view brings great mathematical power. Students should not experience geometry primarily as a way to formalize visual thinking into algebraic or numerical representations. Instead, they should have occasion to gain insight into situations presented numerically or algebraically by transforming them into geometric representations, as well as the more common algebraic or numerical representations of geometric situations. For example, students can use similar triangles to explore questions about integer-coordinate points on a line presented algebraically (Driscoll et al., 2017).

In grades 3–5, students develop many foundational notions of two- and three-dimensional geometry, such as area (including surface area of three-dimensional figures), perimeter, angle measure, and volume. Shape and space work in grades 6–8 is largely about connecting these notions to each other, to students’ lives, and to other areas of mathematics.

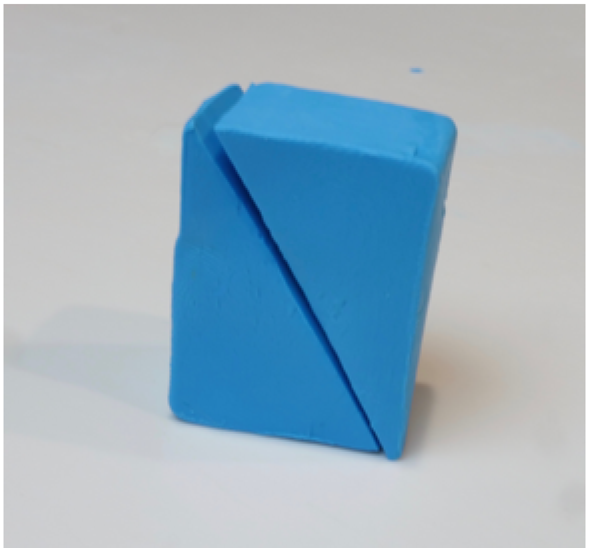
In grade six, for example, two-dimensional and three-dimensional figures are related to each other via nets and surface area (6.G.4), two-dimensional figures are related to algebraic representation via coordinate geometry (6.G.3), and volume is connected to fraction operations by exploring the size of a cube that could completely pack a shoebox with fractional edge lengths (6.G.2). In grade seven, relationships between angle or side measurements of two-dimensional figures and their overall shape (7.G.2), between three-dimensional figures and their two-dimensional slices (7.G.3), between linear and area measurements of two-dimensional figures (7.G.4), and between geometric concepts and real-world contexts (7.G.6) are all important foci.

In grade eight, two important relationships between different plane figures are defined and explored in depth (congruence and similarity), and used as contexts for reasoning in the manner discussed in Chapter 4, the Pythagorean Theorem is developed as a relationship between an angle measure in a triangle and the area measures of three squares (8.G.6). Also, in grade eight, several clusters in the Expressions and Equations domain should sometimes be approached from a geometric point of view, with algebraic representations coming later: In an investigation, proportional relationships between quantities can be first encountered as a graph, leading to natural questions about points of intersection (8.EE.7, 8.EE.8) or the meaning of slope (8.EE.6).

Vignette: Sponge Art

Suzy Dougal, a grade-six teacher, had been wondering about supporting their students' learning with shapes. In previous classes students struggled with 2-D representations of 3-D shapes while they were learning about surface area and volume. Many students had trouble visualizing shapes as they tried to unwrap the faces into a net so they could study surface area.

Figure 7X



Ms. Dougal decided to bring molding clay into class with some clay cutting tools, such as thin wire, fishing line or dental floss. Ms. Dougal began the activity by showing students a rectangular prism they had made out of blue clay (see Figure 7X).

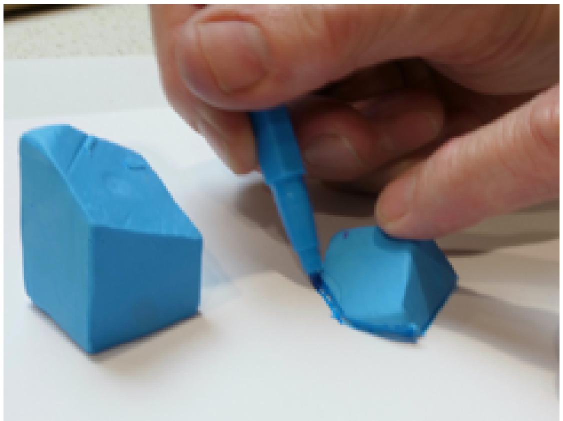
Ms. Dougal asked students to think about cutting the clay prism with one straight cut, thinking about the two shapes that would result from the cut and more specifically the shape of the new faces that are a result of the cut. Students talked in pairs, sharing their ideas about ways to cut the shape and what the two new shapes might look like. Ms. Dougal cut the prism at a diagonal from one short edge through to the other short edge. Ms. Dougal did not separate the two shapes after cutting; instead she asked:

- “What do you think the shape of the new face is?”
- “How many faces does the new shape have?”
- “What are the similarities and differences between the two new shapes?”
- “How is the new face shape similar or different than the shapes of the other faces?”

Students turned and talked to their partners.

After the students discussed their ideas in pairs and shared their ideas with the class Ms. Dougal separated the prism into the two pieces. Ms. Dougal traced the new face on the document camera so students could clearly see the shape of the new face. Ms. Dougal asked, “How accurate were your predictions?” She then asked students, “What different two-dimensional face shapes can you make by slicing a rectangular prism?”

Figure 7X



Ms. Dougal provided students with clay, and a cutting tool (such as thin wire, fishing line or dental floss), isometric and regular dot paper. She asked students to cut the solid they formed to find different shapes that can be made by slicing. For each slice, the group made a sketch of how they cut the solid and traced the sides of the faces to record the new shape they created. Students were asked to record their findings and look for patterns. Students created nets of the original solid and then nets of the two resulting solids following the cut.

For the next phase of the exploration, Ms. Dougal asked students to think about all of the different ways to create shapes from cutting one solid. Students were asked to make these cuts and consider the areas of each face. As they recorded their observations for each new shape they cut, they focused on the resulting face from the cut. Students considered the area of the new face, the surface area of the new shape as well as approximating the volume. For the cut shapes students discussed the patterns they found in their data. Some of the questions Ms. Dougal asked to promote further exploration included:

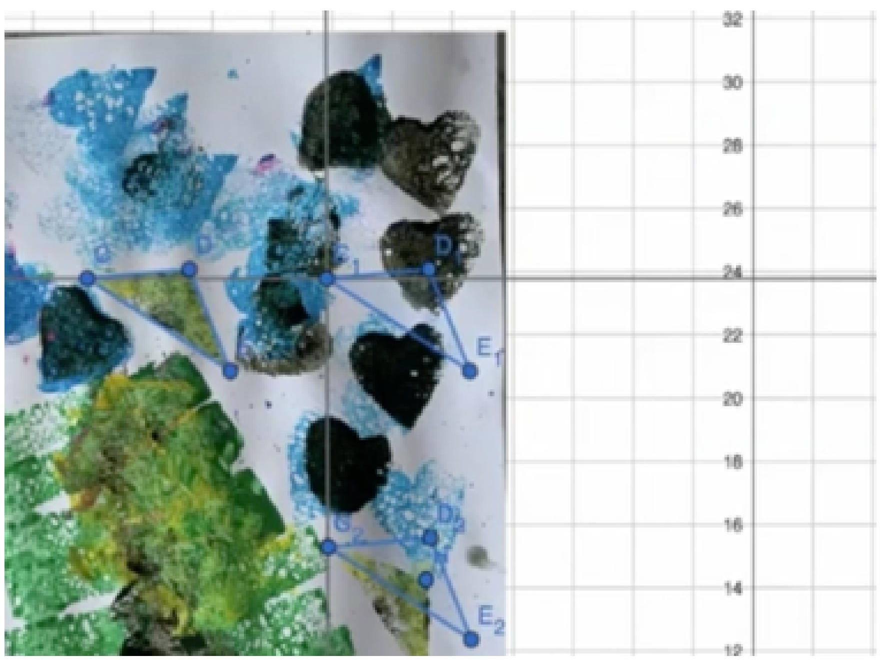
- “How are the nets for the original shape and the new shape similar and different than the original shape?”
- “What data did you collect?”
- “What patterns did you find in your data?”

- “Did you find any patterns between the types of cuts you made?”

Ms. Dougal shared her lesson with her friend Ms. Woodbury. Ms. Woodbury loved the idea and decided to try some adaptations with her sixth-grade class. She asked students to trace the new face image after they had made a cut and her students were given rectangular sponges. The students used paint on the prism faces before and after the cuts to show the different shapes. Students were asked to consider slides, flips, and turns.

Ms. Woodbury connected the activity to geometric transformations and she asked students to upload an image of their sponge painting patterns into Desmos so they could further explore transformations by duplicating two or more of their shapes and then moving them in order to explore the transformation pathways of the shapes. Figure 7X shows the sponge art uploaded into DESMOS.

Figure 7X



Source, and additional detail, are at https://www.youcubed.org/resources/sponge-art-transformations-k-10-video/.

Driver of Investigation 1: Making Sense of the World.

Content Connection: Discovering Shape & Space

CA CCSSM Clusters of Emphasis

- 6.G: Solve real-world and mathematical problems involving area, surface area, and volume.
- 7.G: Draw, construct, and describe geometrical figures and describe the relationship between them. Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.
- 8.EE: Understand the connections between proportional relationships, lines and linear equations. Analyze and solve linear equations and pairs of simultaneous linear equations.
- 8.G: Understand congruence and similarity using physical models, transparencies, or geometry software. Understand and apply the Pythagorean Theorem. Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.

Conclusion

The middle grades of school are critical years when students often decide whether they want to continue or disidentify with mathematics. This chapter outlines a vision for middle school mathematics that engages students in provocative problems through which they may wonder about mathematical relationships and develop curiosity. The mathematical explorations that students encounter can support opportunities for them to appreciate mathematics and include plans for mathematics in their futures. The discussions they can have will allow them to develop self-awareness and to learn to collaborate, as they take on the perspectives of others, and learn important social-emotional skills. Careful discussions of mathematical ideas will also support English learners in learning the language of mathematics. The lack of tracking or acceleration will allow all students to regard mathematics as a subject they can study and in which they belong. This vision for middle years mathematics is organized around Drivers of Investigation, so that all work provides a purpose to understand and explain, predict what could happen, or impact the future. Work should also draw from one of the four content connections—Communicating Stories with data, Taking

Wholes apart, Exploring Changing Quantities, Putting Parts Together, and Discovering Shape and Space, and the Mathematical Practices, as set out in Figure 7X. These big ideas connect different areas of mathematics conceptually and allow students to succeed through engaging tasks and learn mathematics that continues to validate their learning. This chapter has illustrated the kinds of tasks that support learners in this connected, conceptual vision of mathematics learning.

References

Boaler, J. (2011). Changing students' lives through the de-tracking of urban mathematics classrooms. *Journal of urban mathematics education*, *4*(1), 7–14.

Boaler, J. (2015). *Mathematical mindsets: Unleashing students' potential through creative math, inspiring messages and innovative teaching*. John Wiley & Sons.

Boaler, J. (2016). Designing mathematics classes to promote equity and engagement. *The Journal of Mathematical Behavior*, *100*(41), 172–178. http://www.sciencedirect.com/science/article/pii/S0732312315000140.

Boaler, J., Chen, L., Williams, C., & Cordero, M. (2016). Seeing as Understanding: The Importance of Visual Mathematics for our Brain and Learning. Journal of Applied & Computational Mathematics, 5(5), DOI: 10.4172/2168-9679.1000325

Boaler, J. & Foster, D. (2018). Raising Expectations and Achievement. The Impact of Wide Scale Mathematics Reform Giving All Students Access to High Quality Mathematics. https://www.youcubed.org/resources/raising-expectations-achievement-mathematics/.

Boaler, J., & Staples, M. (2008). Creating mathematical futures through an equitable teaching approach: The case of Railside School. *Teachers College Record*, *110*(3), 608–645.

Burris, C. C., Heubert, J. P., & Levin, H. M. (2006). Accelerating mathematics achievement using heterogeneous grouping. *American Educational Research Journal*, *43*(1), 137–154.

Callahan, R. M., Humphries, M., & Buontempo, J. (2020). Making meaning, doing math: high school English learners, student-led discussion, and math tracking. International Multilingual Research Journal, 1–22.

Carpenter, T. P., Gomez, C., Rousseau, C., Steinthorsdottir, O. B., Valentine, C., Wagner, L., et al. (1999, April 19–23). An analysis of student construction of ratio and proportion understanding. Paper presented at the American Educational Research Association, Montreal, Canada.

Cogan, L. S., & Schmidt, W. H. (1999). Middle school math reform. Middle Matters, 8(1), 2–3.

Common Core Standards Writing Team (2019). Progressions for the Common Core State Standards for Mathematics (draft February 7, 2019). Tucson, AZ: Institute for Mathematics and Education, University of Arizona. http://ime.math.arizona.edu/progressions/.

Darling-Hammond, L. (2001). Inequality in teaching and schooling: How opportunity is rationed to students of color in America. BD Smedley, AY Stith, L. Colburn, C. & H. Evans (Eds.), The right thing to do—The smart thing to do, 208–233.

Domina, T., McEachin, A., Penner, A., & Penner, E. (2015). Aiming High and Falling Short: California's Eighth-Grade Algebra-for-All Effort. Educational Evaluation and Policy Analysis, 37(3), 275–295. https://doi.org/10.3102/0162373714543685.

Driscoll, M. J., DiMatteo, R. W., Nikula, J., & Egan, M. (2007). *Fostering geometric thinking: A guide for teachers, grades 5–10*. Portsmouth, NH: Heinemann.

Falco, L. D. (2019). An intervention to support mathematics self-efficacy in middle school. *Middle School Journal*, *50*(2), 28–44.

Freeman, Y. S., & Freeman, D. E. (2002). Closing the Achievement Gap: How to Reach Limited-Formal-Schooling and Long-Term English Learners. Portsmouth, NH: Heinemann.

Inside Mathematics (n.d.). *Formative Re-Engaging Lessons*. University of Texas at Austin, Charles A. Dana Center. Retrieved October 9, 2020 from https://www.insidemathematics.org/classroom-videos/formative-re-engaging-lessons.

Lawyers' Committee for Civil Rights of the San Francisco Bay Area. (2013). *HELD BACK: Addressing Misplacement of 9th Grade Students in Bay Area School Math Classes*. https://lccr.com/wp-content/uploads/HELD-BACK-9th-Grade-Math-Misplacement.pdf.

Lamon, S. J. (1993). Ratio and proportion: Connecting content and children's thinking. Journal for research in mathematics education, 41–61.

Lamon, S. J. (2012). Changing Instruction. In Teaching Fractions and Ratios for Understanding: Essential Content Knowledge and Instructional Strategies for Teachers. Routledge.

Loveless, T. (2013). The resurgence of ability grouping and persistence of tracking. Part II of the 2013 Brown Center Report on American Education, the Brookings Institution. [2], 1.

National Council of Teachers of Mathematics (NCTM). 2020. *Catalyzing Change in Middle School Mathematics: Initiating Critical Conversations*. Reston, VA: NCTM.

Moschkovich, J. (2013). *Principles for Mathematics Instruction for ELLs*. Retrieved from Stanford University, UL/SCALE website: https://ell.stanford.edu/teaching_resources/math.

Pajares, F., & Graham, L. (1999). Self-efficacy, motivation constructs, and mathematics performance of entering middle school students. *Contemporary educational psychology*, *24*(2), 124–139.

Petersen, J. L., & Hyde, J. S. (2017). Trajectories of self-perceived math ability, utility value and interest across middle school as predictors of high school math performance. *Educational Psychology*, *37*(4), 438–456.

Progressions: http://mathematicalmusings.org/wp-content/uploads/2018/08/ccss_progression_nf_35_2018_08_10.pdf

San Francisco Unified School District Mathematics Department (2015). *Re-engagement*. Retrieved from San Francisco Unified School District website: http://www.sfusdmath.org/reengagement.html.

Steinthorsdottir, O. B., & Sriraman, B. (2009). Icelandic 5th-grade girls' developmental trajectories in proportional reasoning. *Mathematics Education Research Journal*, *21*(1), 6–30.

Walqui, A., & Van Lier, L. (2010). *Scaffolding the Academic Success of Adolescent*

English Language Learners: A Pedagogy of Promise. WestEd. 730 Harrison Street, San Francisco, CA 94107-1242.

Williams, K. L., Burt, B. A., & Hilton, A. A. (2016). Math achievement: A role strain and adaptation approach. *Journal for Multicultural Education 10*(3), 368–383.

Zwiers, J., Dieckmann, J., Rutherford-Quach, S., Daro, V., Skarin, R., Weiss, S., & Malamut, J. (2017). Principles for the design of mathematics curricula: Promoting language and content development. Retrieved from Stanford University, UL/SCALE website: http://ell.stanford.edu/content/mathematics-resources-additional-resources.

Zwiers, J. (2018). Developing Reasoning and its Language in Secondary Mathematics Instruction. *Soleado—Promising Practices from the Field, 11*(1), 1–11. Retrieved from https://www.dlenm.org/wp-content/uploads/2019/11/Soleado_Fall_2018_Final_8_17.pdf.