Mathematics Framework
Chapter 2: Teaching for Equity and Engagement

Second Field Review Draft

Equity and Engagement: An Introduction

In California, all teachers strive to ensure every child has an equitable opportunity to succeed. Teachers of mathematics can work to ensure that all students receive the attention, respect, and resources they need to achieve success. A long-standing body of research in the fields of education and psychology shows that students learn best through active engagement (Bransford et al., 2005) with mathematics and one another (Freeman et al., 2014).

Real-world tasks can offer students opportunities to mathematize contexts that connect to their lived experiences. When teachers get to know their students—not only how they think about mathematics, but also their curiosities, interests, and experiences—they are better able to choose, craft, and launch tasks that engage students with big ideas in meaningful and relevant ways. As such, this framework highlights the importance of students' active engagement in classrooms through mathematical inquiry and investigation. When teachers launch investigations into relevant content with the Drivers of Investigation (DI) identified in this framework, they elicit students' curiosity and provide motivation for them to engage deeply with authentic mathematics. The framework suggests the following Drivers of Investigation:

1) Make sense of the world
2) Predict what could happen
3) Impact the future

Research conducted in preceding decades has produced a wealth of information showing that mathematics learning, understanding, and enjoyment comes **when students are actively engaged with mathematical concepts**—when they are developing mathematical curiosity, asking their own questions, reasoning with others, and encountering mathematical ideas in multi-dimensional ways. This can occur through engagement with numbers, but also through visuals, words, movement, and objects, considering the connections between them (Boaler, 2016, 2019; Cabana, Shreve, and Woodbury, 2014; Louie, 2017; Hand, 2014; Schoenfeld, 2002). The principles of the Universal Design for Learning (UDL) guidelines outline a multi-dimensional guide that benefits all students, and can be particularly useful when applied to mathematics.

When students are engaged in these kinds of experiences, they can come to view mathematics, and their own relationship to mathematics, far more positively. The contrasting approach—of students sitting in rows watching a teacher demonstrate methods before reproducing them in short exercise questions unconnected to real data or situations —has led to widespread mathematical disinterest, perpetuating the common perspective that mathematics is merely a sterile set of rules. This form of learning mathematics has been challenged for decades; for example, the 1973 publication of the case of Benny (Erlwanger, 1973) – a student who was able to perform computational tasks while generating a stunning set of misconceptions to explain them – demonstrated the importance of learning with understanding and, in particular, of creating opportunities for students to actively make sense of mathematical concepts and strategies.

Students benefit from viewing mathematics as a vibrant, interconnected, beautiful, relevant, and creative set of ideas. As educators create opportunities for students to engage with and thrive in mathematics and value the different ways questions and problems can be approached and learned, many more students view themselves as belonging to the mathematics community (Boaler and Staples, 2008; Boaler, 2016; Langer-Osuna, 2014; Walton et al., 2012). Such an approach prepares more students to think mathematically in their everyday lives, and helps society develop many more students interested in and excited by Science, Technology, Engineering, Arts, and Mathematics (STEAM) pathways.

California’s diverse student population brings to schools a broad range of interests, experiences, and linguistic and cultural assets. Cultural and personal relevance is important for learning and also for creating mathematical communities that reflect California’s diversity. Educators can learn to notice, utilize, and value students’ identities, assets, and cultural resources to support learning for all students in California. In this chapter, there are several examples of such teaching in action, giving particular attention to a range of learning needs, including supports for language development, learning differences, and high achievement, all of which can exist together. Students can, of course be a high achieving language learner or be high achieving with learning differences. This framework also offers ideas for teaching in ways that promote racial justice and create space for students with a wide range of social identities to feel a sense of belonging *as they are* to the mathematics community.

Linguistic diversity is a key feature of California and relevant to the teaching and learning of mathematics (Moschkovitch, 1999, 2009, 2014). Various supports exist to ensure that the state’s large population of language learners and multilingual students can learn and thrive. These supports reflect important recommendations for students learning English—for example Moschkovitch (2014), Lagunoff et al. (2015), and Turner et al. (2013), as well as developing culturally relevant lessons (Ladson-Billings, 2009; Hammond, 2020; Milner, 2011). These recommendations focus on different ways of giving all students access to meaningful mathematics. A framework outlined by Darling (2019) seems particularly important in encouraging linguistically and culturally diverse language learners, as well as other students:

1. Take an **asset approach** and recognize multilingualism as a power

2. **Include group work** (strategically grouping for language development)
3. **Make work visual** (include graphic organizers, visual examples, encourage visual communication)
4. Build on students' lived experiences and cultures (allow native and home language use)
5. **Scaffold learning and language development** (including sentence frames and/or sentence starters and many other supports)
6. **Give opportunities for pre-learning** (opportunities to learn prerequisite material ahead of time)

Through Darling’s (2019) framework, teachers can productively analyze their own lesson plans and consider ways to draw upon the rich cultural and linguistic resources of their students (Nasir et al., 2014; Fernandez, 2017; Louie, 2017). In the sections that follow, we draw from California’s English Language Development Standards (ELD Standards) (California Department of Education, 2012), the California Department of Education’s advice for integrating the ELD Standards into mathematics teaching (California Department of Education, 2021a), the principles of UDL (CAST, 2018), and the California Department of Education’s advice for asset-based pedagogies (California Department of Education, 2021b.)

The following vignette demonstrates an open-ended task that all students can access, and that extends to sufficient depth that all students remain challenged (that is, a “low floor, high ceiling” task), and the use of innovative learning models that can map out the pathways students most need.

Vignette: A Personalized Learning Approach

Spring Hill Middle School will partner with an innovative learning model provider to implement a unique and personalized approach to mathematics that enables each student to progress on his or her own learning path. The model integrates a combination of teacher-led, collaborative, and independent learning modalities in ways that enable students to build deep conceptual understanding and apply their learnings in real-world contexts.

At the start of the year, each student in a cohort will take a diagnostic assessment, the data from which is used to build a personalized set of mathematical ideas that the student will learn for the year. This will help the students, their teachers and their parents to understand what the focus of their learning will be for the year and why.

Each student's set of ideas will be different, but could include some below grade level concepts that the student either didn't learn the previous year or forgot over the summer, as well as some ideas aligned to seventh grade standards and could also include concepts that they otherwise would not learn until eighth grade or integrated high school courses.

Each student's progress through this set of ideas is made visible using advanced technology that allows students, teachers, and parents to see a snapshot of how a student is doing at any given time. This technology is able to take stock of the needs of the entire class of students and assign a low floor, high ceiling project that everyone in the cohort can engage with. In this example, the project is focused on decomposing shapes to find their area. Over the time that students are engaging in this project, they will also experience shorter lessons on related mathematical ideas that will best support their growth, regardless of what mathematics they know going in.

At the same time, students will also engage in their own personalized schedule of lessons. In these lessons, students will explore related concepts through a variety of modalities. Some of the time they will learn in a large group from a teacher, some of the time they will collaborate with peers on a novel problem, and some of the time they will learn independently. This learning can support and extend the understandings students are building in the project.

The Students

Monique is a currently high achieving sixth grade student who is ready to learn a new sixth grade geometry concept. Over the course of a few weeks, she will work on a project with a heterogeneous group of her peers to make connections between finding the area of a rectangle and calculating the area of new and more complex shapes.

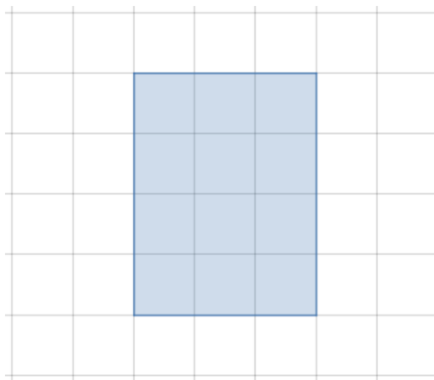
Darren is another student in Monique's class. He is less experienced in geometry than Monique, but will be able to engage in the same project as Monique because it is accessible at many levels. The task is open enough that Darren is able to utilize his knowledge of sketching to visually explore the task shapes in ways that allow him to reason through possible solution strategies. The project will provide Darren access to supports to the grade level standards and also the grade level standards themselves.

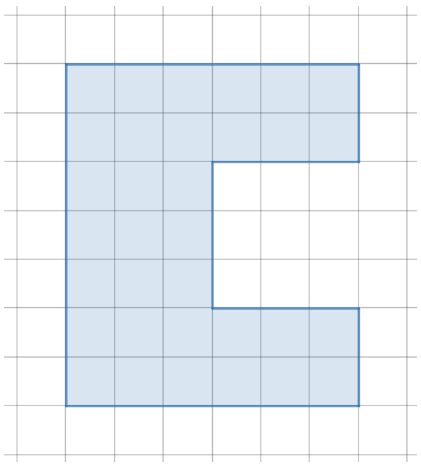
Project

Adapted from Boaler, Munson, and Williams (2018).

In this project, students will explore making art out of polygons on grids. They will use the grids to explore and find ways to determine area through decomposition. They will develop strategies that always work for finding the area of familiar figures and employ those strategies to find the area of any polygon on a grid.

- CA CSS 3.MD.7a- Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths.
- CA CSS 3.MD.7d - Recognize area as additive. Find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts, applying this technique to solve real-world problems.
- CA CSS 6.G.1- Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

Slide	Project Description
Slide 4 	**Goal:** Students recognize the need for area to measure the size of a rectangle, given that it has two dimensions. Possible Teacher Moves: ● Ask students how they would describe this rectangle. If they do not naturally start talking about area or how the rectangle is made up of little squares, it might help to redirect them by asking about the size of the rectangle. You could even draw another rectangle that is a different shape and ask how they would compare the sizes. ● After students have made their observations, define the area of the rectangle as the space occupied by a two-dimensional shape. "You can find the area of this rectangle by counting unit squares or multiplying the dimensions." Possible Student Moves: ● "It's a 3×4 rectangle." ● "It's made up of 12 squares." ● "It's perimeter is 14 units." ● "It's area is 12 square units. I know that because 3×4=12."

Slide	Project Description
Slide 5 	**Goal:** Students think about how to measure the size of a polygon given that it has two dimensions. Possible Teacher Moves: ● Ask students to find the area of the shaded figure. ● Give them time to think. This could happen individually or in pairs. ● Highlight different approaches that students take. Possible Student Moves: ● Students could count the squares, break the shape into 3 rectangles in multiple different ways or even find the area of the larger 6×7 rectangle and subtract out the 3×3 square. It is important that students see that all of these strategies are equally valid and come to the same conclusion.

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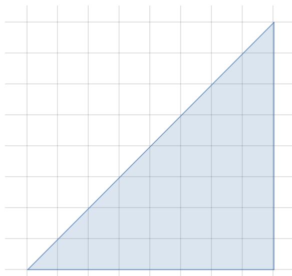
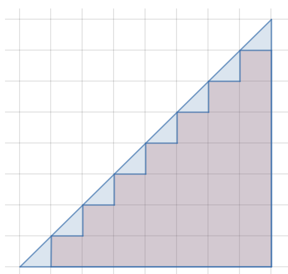
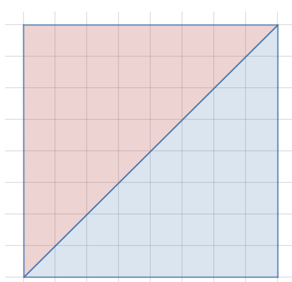
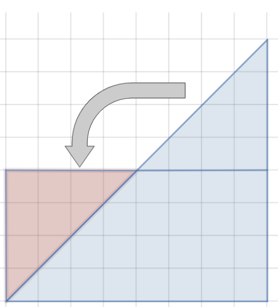
Slide	Project Description
Slides 6-9    	**Goal:** Students develop strategies for making sense of partial squares when finding area. Possible Teacher Moves: ● Show students slide 5 and ask them to explore different ways of finding the area of the triangle. ● Give students time to think. This could be done individually or in a pair. ● Highlight different approaches students take. Some of their approaches might match one of the images on slides 6-8. If so, they can be used to help share that strategy. ● Share the images on slides 6-8 and ask students how each of the images can help them think about the area of the triangle. Possible Student Moves: ● Students will think about how to do this in a variety of ways. Some possibilities are represented visually on slides 6-8, but they might have other approaches too. If they do, encourage them to share their thinking visually. ● Slide 6 - Students might count or otherwise calculate the number of whole squares first, and then move on to the partial squares. They could count these as halves, or pair them up to make wholes. ● Slide 7 - Students might recognize that 2 of the triangles make a square, and thus the area of 1 triangle must be half the area of the square. ● Slide 8 - Students might notice that the top portion of the triangle can be rotated down to make a rectangle.

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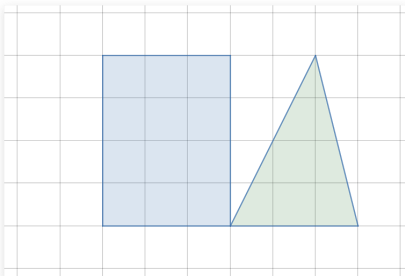
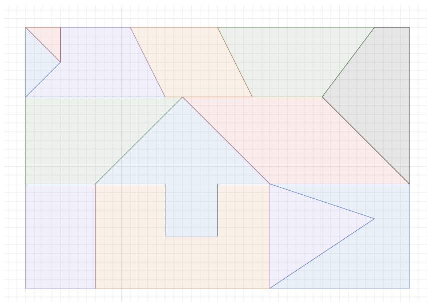
Slide	Project Description
Slide 10 	**Goal:** Students recognize that a triangle has half the area of a rectangle with the same base and height. Possible Teacher Moves: ● Ask students how the area of this rectangle and triangle compare. ● It is important to note that the partial squares in this example are not half squares. If students seem to have a misunderstanding about this point, it might help to bring the class together to highlight it. ● It might be helpful to use **the original construction** so that you can move the pieces around, draw lines, etc. Possible Student Moves: ● Some students might recognize that the base and height of the rectangle and triangle are the same, and therefore conclude that the areas are either (a) the same or (b) the area of the rectangle is double the area of the triangle. Press these students to show this relationship visually. ● Some students will find the area of each figure to compare their areas. They could do this by counting the full squares and then matching up the partial squares in the triangle to make two 1×2 rectangles. ● Some students may show that, when the triangle is placed on top of the rectangle with the bases aligned, and a line is drawn straight down through the apex of the triangle, it forms 2 pairs of right triangles with the same dimensions and therefore the same area.

Slide	Project Description
Slide 11 	**Goal:** Students apply their thinking about finding the area of figures to shapes at various levels of challenge. Possible Teacher Moves: ● Share this slide. Ask students what shapes they recognize. This is a good opportunity to get a sense for what types of figures students are familiar with. ● Provide the image as a handout to students and ask them to choose 3 shapes in it to find their area. ● Explain that students will be creating their own grid art as their final project for this deep dive, and it could look something like this or could look different. ● Choose a few students who used different strategies and ask them to share their thinking with the class. Help the class make connections among the different strategies. Possible Student Moves: ● Students may choose more familiar or less familiar figures. Ask them to share their thinking about why they chose the shapes they did and how they found the areas.

Monique's Experience

Monique started the project with the understanding that shapes can sometimes be broken into smaller rectangles. Through her project work, she is able to extend that big idea to see that shapes can be broken into triangles as well. She is able to make connections between rectangles and triangles and think flexibly about 2D figures. Monique is able to break down the image on slide 11 into different types of triangles and quadrilaterals to create a new image incorporating all different shapes. Using daily exit slip data, algorithms are able to pinpoint that Monique is ready to extend her knowledge past the 6th grade geometry concept and move onto CA CSS 7.G.1. (Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.)

The next day, Monique works independently on a virtual lesson that includes instructional videos, Geogebra applets, and interactive practice problems. For example, Monique explores an applet at https://www.geogebra.org/m/WFbyhq9d to see that triangles can be found inside of circles and is able to apply her learning using an interactive platform.

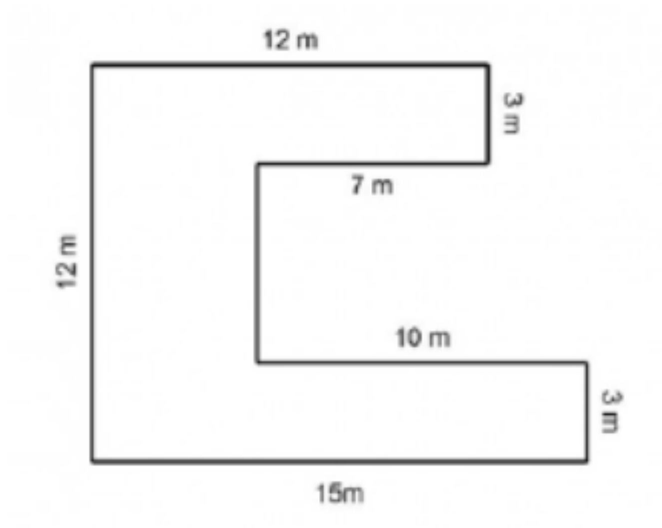
During this non-project time, Monique is working on above grade level skills independently while there are various personalized lessons happening in different cohorts. After this lesson, Monique will have the opportunity to work in a small group with other students who are working on the same concept and teachers will be able to monitor her progress by reviewing her exit slips at the end of the day and checking in with her doing a daily advisory session.

Monique began this project ready to learn the on-grade level concept. Through this low floor high ceiling task, she was able to intuitively learn new concepts and collaborate with peers who were also learning at their own pace. Through the personalized lesson, she was able to extend her knowledge of decomposing shapes to triangles, polygons and even circles; an important understanding for geometry and even calculus.

Darren's Experience

Coming into the project, Darren is comfortable with the idea of finding the area of a rectangle, but has not had much experience with decomposing shapes to find their area. Having the opportunity in this project to connect the arduous task of counting all the squares in a figure composed of rectangles (as in slide 5) to seeing that it can be broken into rectangles with the support of the grid, supports Darren in thinking about finding the area of these shapes in more sophisticated ways.

Later, in a non-project session, Darren has a conversation with a partner about a similar problem, this time without the support of the grid. They are charged with individually finding the area of this figure:



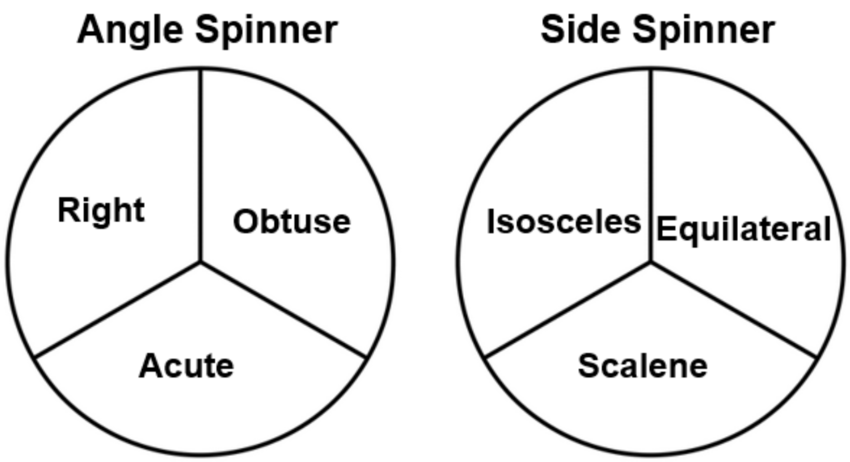
When they have both spent a few minutes working on this, they discuss the following questions:

- Did the problem provide you with all the information you needed to find the area or did you need to figure some values out first?
- Compare the way you broke up the shape with the way your partner did it. Did you use the same strategy? Were there other strategies you could have used?
- What is one mistake that someone might make when trying to find the area of complex shapes like this?

Discussing these questions with a partner helps Darren to see that there are many different ways to find the areas of these figures, and they all produce the same answer if they are done correctly. It also helps him to identify and put a name to the common error of breaking a figure into shapes that overlap and thus calculating the area of some parts of the figure twice.

All of this work on decomposing rectangular shapes lays a foundation for him to start to study the area of triangles, and extend that thinking to now develop strategies for finding the area of partial squares on the grid. But, before he does that, he experiences a lesson led by a teacher in another non-project session where he explores the different classifications of triangles. (CA CSS 5.G.3 - Classify two-dimensional figures in a hierarchy based on properties.) Understanding these classifications will enable Darren to thoroughly explore the different types of possible triangles to ensure that any patterns he sees in finding triangle areas are universal.

In this lesson, after defining the key terms, the teacher breaks the students into small groups to play a game. Students are provided with the following spinners:



They take turns spinning both spinners and trying to draw a triangle that satisfies both conditions, making note of which triangles are possible or impossible and why. By doing this, they become familiar with the different types of triangles and learn the constraints on creating them.

That sets the stage for Darren to explore the area of triangles. This poses a challenge in that he can no longer think solely in terms of whole squares on a grid. Reasoning about the different strategies for finding the area of the triangle enables him to make sense of, and connections among, a few different approaches. Later, when he comes across less regular triangles and other shapes, he is able to apply his thinking about different ways to decompose a figure to develop a problem solving strategy. His understanding continues to be supported by non-project lessons that connect to and extend the thinking he's done in the project.

Darren began this project below grade level in this area of geometry and not prepared to dive right into the grade level standards. By engaging in a project with multiple access points, supported by a personalized schedule of large group, small group and individual lessons, he was able to make up significant ground. What's more, because he engaged with the mathematical ideas involved at a conceptual level through the lens of decomposing shapes, he is poised to do further learning in the future.

Multi-Dimensional Tasks

Research in the fields of education, psychology, and neuroscience has revealed important knowledge on neurodiversity and its implications for teaching and learning. Studies show that students with identified learning differences are supported in mathematics classrooms when they include multi-dimensional tasks that incorporate multiple representations, multiple ways to engage, and multiple forms of expression (Foote and Lambert, 2011; Lambert and Sugita, 2016; Moschkovich, 1999; Boaler and LaMar, 2019). Drawing from the work of Rachel Lambert and others, the following strategies support the participation of students with identified learning differences in mathematical discussions:

- Including paraprofessionals in the instruction allows students opportunities to rehearse and share in preparation for whole-class discussion (Baxter et al., 2005). This functions similarly to a think-pair-share completed prior to whole-class discussion.
- Teachers can help create a classroom culture where all students can and *do* readily access resources—like math notebooks, media apps and websites, and manipulatives—whenever they need them. Some students may use particular resources more often or for longer amounts of time than other students during whole class discussions and benefit from being able to draw on them as necessary (Foote and Lambert, 2011).
- Asking follow-up questions sets up the expectation and the support for students to be accountable to explaining their strategies. (Lambert and Sugita, 2016).

Strategies that support students with identified learning differences ultimately create a positive learning environment for all students. Incorporating gestures (including facial and corporal), artifacts (such as props and images), and multiple styles of language as part of classroom discourse (include formal and informal dialects, AAVE, Code Switching, and Translanguaging, etc.), attending to and revoicing students' mathematical ideas, allowing students time to rehearse and prepare for whole-class discussions, not limiting the use of resources, and using follow-up questions to help students complete or extend their explanations support the participation of all learners. Such strategies are particularly helpful for language learners.

Five Components of Equitable and Engaging Teaching for All Students

“Creating, supporting, and sustaining a culture of access and equity require being responsive to students' backgrounds, experiences, cultural perspectives, traditions, and knowledge when designing and implementing a mathematics program and assessing its effectiveness. Acknowledging and addressing factors that contribute to differential outcomes among groups of students are critical to ensuring that all students routinely have opportunities to experience high-quality mathematics instruction, learn challenging mathematics content, and receive the support necessary to be successful. Addressing equity and access includes both ensuring that all students attain mathematics proficiency and increasing the numbers of students from racial, ethnic, linguistic, gender, and socioeconomic groups who attain the highest levels of mathematics achievement.”

–National Council of Teachers of Mathematics (NCTM) Position Statement, Access and Equity in Mathematics Education

How does a teacher create an equitable and engaging mathematics environment that supports all students, from students with little experience in a mathematical practice or content area to those who are already proficient, from those who are just learning English to those who are native speakers, and for all students who learn in a wide variety of ways? The following sections describe five important components that are based on research and supported by practice. The Five Components of Equitable and Engaging Teaching are: 1) Plan Teaching Around Big Ideas; 2) Use Open, Engaging Tasks; 3) Teach Toward Social Justice; 4) Invite Student Questions and Conjectures; 5) Prioritizing Reasoning and Justification. These ideas are aligned with other important resources, such as the Teaching for Robust Understanding (TRU) Framework (TRU Framework, 2018) and the *Access and Equity: Promoting High Quality Access* Series from NCTM. Books such as *The Impact of Identity in K-8 Mathematics* (by Julia Aguirre, Karen Mayfield and Danny B Martin), *Teaching Math to Multilingual Learners* (by Kathryn Chavl), and *Teaching Math to English Learners* (by Debra Coggins) are also helpful.

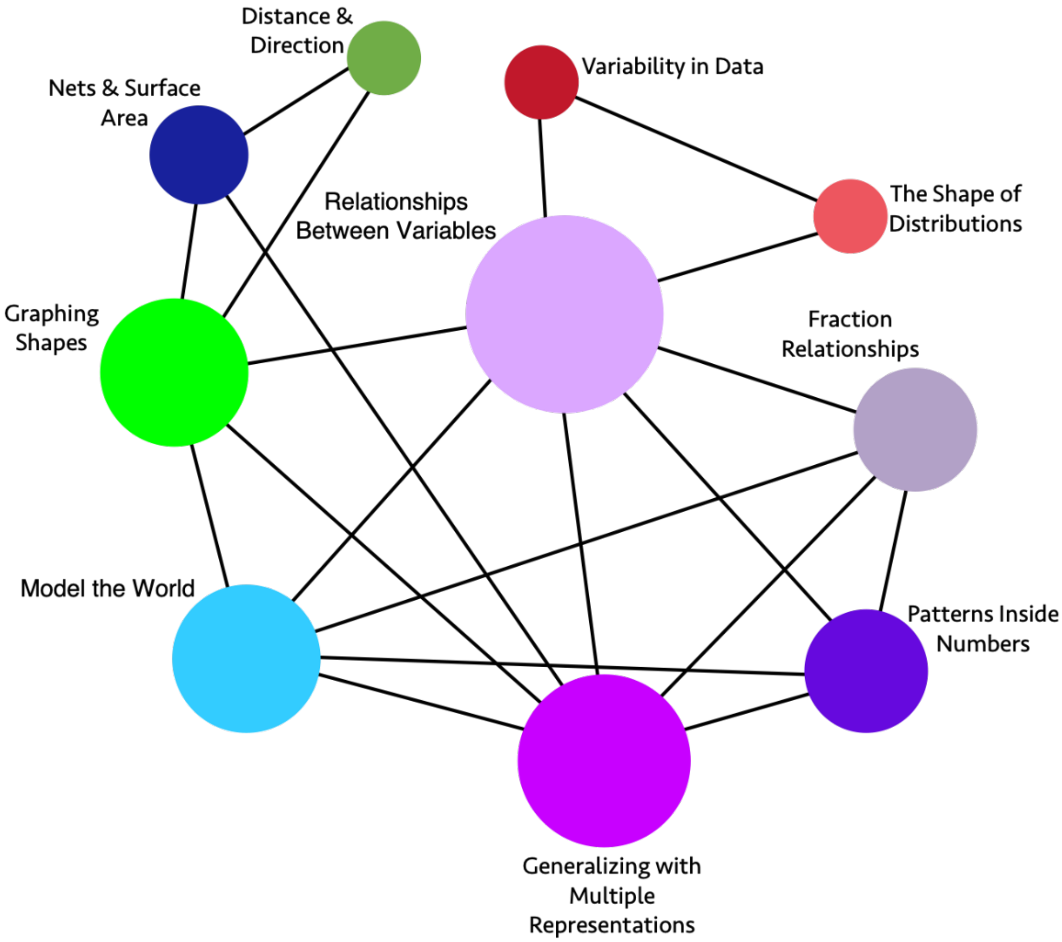
Plan Teaching Around Big Ideas

Mathematics is a subject made up of important ideas and connections. Curriculum standards tend to divide the subject into smaller topics, but it is important for teachers and students to think about the big ideas that characterize mathematics at their grade levels and the connections between them. Instead of planning teaching around the small topics or methods set out in the standards, or the chapters of textbooks, teachers can plan to teach the "big ideas" of mathematics (Nasir et al., 2014). Big ideas are especially important in driving meaningful and coherent curriculum, in which lessons develop important content and mathematical practices in the context of authentic student investigations.

Although various big ideas are present in TK–12 mathematics, and many teachers may themselves envision different major themes in the standards, this framework sets forth the notion of "big idea" teaching in two important ways. First, in terms of lesson design, the Standards for Mathematical Practice, the four Content Connections, and the three Drivers of Investigation, can be connected to form the How, What and Why of a lesson or activity. The SMPs describe the How (how students engage), the (CCs) describe What (what overarching topics and connections will be learned, see below for content big ideas), and DIs provide the Why (why this mathematics is relevant, in a bigger sense).

Secondly, the Content Connections are phrased to be very broad, as they span TK–12. So a focused set of big ideas, organized by grade level and CA CCSSM content standards, was created as part of the California Digital Learning Integration and Standards Guidance initiative (California Department of Education, 2021c). These grade level big ideas, organized by Content Connections, and inclusive of multiple CA CCSSM content standards, are presented in Appendix A for grades TK–10. In that Appendix, a table of Content Connections, focused into big ideas for each grade level appears first. Next, a network map of the big ideas (circular nodes) and connections between them (line segments) for each grade level follows. Each network diagram is followed by a table indicating the CC and the relevant content standards for each big idea. For the diagrams, it should be noted that the sizes of the nodes are related to the number of connections to standard, for each. Figures 2.1 and 2.2 are examples from Grade 6, shown below.

Figure 2.1. Grade 6 Map of Big Ideas



[Link to long description]

Figure 2.2. Grade 6 Content Connections, Big Ideas, and Standards

Content Connection	Big Idea	Grade 6 Standards
Communicating Stories with Data	**Variability in Data**	**SP.1, SP.5, SP.4:** Investigate real world data sources, ask questions of data, start to **u**nderstand variability - within data sets and across different forms of data, consider different types of data, and represent data with different representations.

Content Connection	Big Idea	Grade 6 Standards
Communicating Stories with Data	**The Shape of Distributions**	**SP.2, SP.3, SP.5:** Consider the distribution of data sets - look at their shape and consider measures of center and variability to describe the data and the situation which is being investigated.
Exploring Changing Quantities	**Fraction Relationships**	**NS.1, RP.1, RP.3:** Understand fractions divided by fractions, thinking about them in different ways (e.g., how many 1/3 are inside 2/3?), considering the relationship between the numerator and denominator, using different strategies and visuals. Relate fractions to ratios and percentages.
Exploring Changing Quantities	**Patterns inside Numbers**	**NS.4, RP.3:** Consider how numbers are made up, exploring factors and multiples, visually and numerically.
Exploring Changing Quantities	**Generalizing with Multiple Representations**	**EE.6, EE.2, EE.7, EE.3, EE.4, RP.1, RP.2, RP.3:** Generalize from growth or decay patterns, leading to an understanding of variables. Understand that a variable can represent a changing quantity or an unknown number. Analyze a mathematical situation that can be seen and solved in different ways and that leads to multiple representations and equivalent expressions. Where appropriate in solving problems, use unit rates.
Exploring Changing Quantities	**Relationships Between Variables**	**EE.9, EE.5, RP.1, RP.2, RP.3, NS.8, SP.1, SP.2:** Use independent and dependent variables to represent how a situation changes over time, recognizing unit rates when it is a linear relationship. Illustrate the relationship using tables, 4 quadrant graphs and equations, and understand the relationships between the different representations and what each one communicates.
Taking Wholes Apart, Putting Parts Together	**Model the World**	**NS.3, NS.2, NS.8, RP.1, RP.2, RP.3:** Solve and model real world problems. Add, subtract, multiply, and divide multi-digit numbers and decimals, in real-world and mathematical problems - with sense making and understanding, using visual models and algorithms.

Content Connection	Big Idea	Grade 6 Standards
Taking Wholes Apart, Putting Parts Together & Discovering Shape and Space	**Nets and Surface Area**	**EE.1**, **EE.2, G.4, G.1, G.2, G.3:** Build and decompose 3-D figures using nets to find surface area. Represent volume and area as expressions involving whole number exponents.
Discovering Shape and Space	**Distance and Direction**	**NS.5, NS.6, NS.7, G.1, G.2, G.3, G.4:** Students experience absolute value on numbers lines and relate it to distance, describing relationships, such as order between numbers using inequality statements.
Discovering Shape and Space	**Graphing Shapes**	**G.3, G.1, G.4, NS.8, EE.2:** Use coordinates to represent the vertices of polygons, graph the shapes on the coordinate plane, and determine side lengths, perimeter, and area.

NCTM's (2014) Principles to Action also offers guidance to plan goals around learning big ideas, and links these with student beliefs:

NCTM, Principles to Action, 2014

Beliefs About Teaching and Learning Mathematics

Unproductive beliefs	Productive beliefs
Mathematics learning should focus on practicing procedures and memorizing basic number combinations.	Mathematics learning should focus on developing understanding of concepts and procedures through problem solving, reasoning, and discourse.
Students need only to learn and use the same standard computational algorithms and the same prescribed methods to solve algebraic problems.	All students need to have a range of strategies and approaches from which to choose in solving problems, including, but not limited to, general methods, standard algorithms, and procedures.
Students can learn to apply mathematics only after they have mastered the basic skills.	Students can learn mathematics through exploring and solving contextual and mathematical problems.

Unproductive beliefs	Productive beliefs
The role of the teacher is to tell students exactly what definitions, formulas, and rules they should know and demonstrate how to use this information to solve mathematics problems.	The role of the teacher is to engage students in tasks that promote reasoning and problem solving and facilitate discourse that moves students toward shared understanding of mathematics.
The role of the student is to memorize information that is presented and then use it to solve routine problems on homework, quizzes, and tests.	The role of the student is to be actively involved in making sense of mathematics tasks by using varied strategies and representations, justifying solutions, making connections to prior knowledge or familiar contexts and experiences, and considering the reasoning of others.
An effective teacher makes the mathematics easy for students by guiding them step by step through problem solving to ensure that they are not frustrated or confused.	An effective teacher provides students with appropriate challenge, encourages perseverance in solving problems, and supports productive struggle in learning mathematics.

It is helpful if mathematics teachers are given release time in which they can sit with colleagues and discuss the big ideas in their grade level or course, then choose rich, deep tasks that invite students to explore and grapple with those big ideas (Arbaugh and Brown, 2005). The Big Idea diagrams and descriptions in Appendix A are an important resource for this work. The cluster headings that organize the standards in the Common Core State Standards for Mathematics (CA CCSSM) also give a broader view of mathematical ideas than the detail of individual standards and can usefully form a guide for such discussions and choosing of tasks. These tasks can then form the basis of a course and, if the tasks are rich enough, they likely include many of the smaller methods and ideas set out in the standards (Smith and Stein, 2011). Further, reinforcements – that is, revisiting ideas previously learned- can easily be built into investigations or challenges in the later grades. Rather than preparing a set of many problems to work through in a lesson, one rich task may be planned as the basis for an entire lesson, or at times worked on through several days of exploration, sense-making, and discussion. More detail is given on these kinds of tasks in section two.

There are times when teachers will share ways to approach a mathematics problem or discuss with students new methods to learn important mathematical concepts. Important research studies have considered the best times to enact such direct instruction (Schwartz and Bransford, 1998; Deslauriers et al., 2019). The studies contrasted the approach used in many classrooms—a practice of teaching students the methods and then providing opportunities to practice those methods—with a different approach, one where teachers introduced questions first, then allowed students time to use intuition in considering ways they may approach the questions. In these studies, teachers taught new methods to students when they needed them to solve problems (NCTM, 2014). The students who learned through this approach achieved at significantly higher levels, leading the researchers to conclude that their understanding came because their brains were primed to learn the methods—methods they knew they needed in order to solve the problems—and so they were engaged and interested. An illustration of this approach is provided in the next vignette. Students also engaged in struggle, which is the most productive part of learning. When students learn methods before they use them, they might ask the legitimate question, *When will I ever need this?* The vignette below describes a high school classroom in which the teacher taught mathematical methods when students needed them to solve the problem. In working on this task, the students received opportunities to learn.

Vignette–36 Fences

Lori, a high school geometry teacher, introduces a problem to students at the start of a 90-minute class period. Lori explains that a farmer has 36 individual fences, each measuring one meter in length, and that the farmer wants to put them together to make the biggest possible area. Lori takes time to ask her students about their knowledge of farming, making reference to California's role in the production of fruit, vegetables, and livestock. The students engage in an animated discussion about farms and the reasons a farmer may want a fenced area. While some of Lori's long-term English learners show fluency with social/conversational English, she knows some will be challenged by forthcoming disciplinary literacy tasks. To support meaningful engagement in increasingly rigorous course work, she ensures images of all regular and irregular shapes are posted and labeled on the board, along with an optional sentence frame, "*The fence should be arranged in a* [blank] *shape because* [blank]*."* These support instruction when Lori asks students what shapes they think the fences could be arranged to form. Students suggest a rectangle, triangle, or square. With each response, Lori reinforces the word with the shape by pointing at the image of the shapes. When she asks, "How about a pentagon?" she reminds students of the optional sentence frame as they craft their response. Lori asks the students to think about this and talk about it as mathematicians. Lori asks them whether they want to make irregular shapes allowable or not.

After some discussion, Lori asks the students to think about the biggest possible area that the fences can make. Some students begin by investigating different sizes of rectangles and squares, some plot graphs to investigate how areas change with different side lengths.

Susan works alone, investigating hexagons—she works out the area of a regular hexagon by dividing it into six triangles and she has drawn one of the triangles separately. She tells Lori that she knew that the angle at the top of each triangle must be 60 degrees, so she could draw the triangles exactly to scale using compasses and find the area by measuring the height.

Niko finds that the biggest area for a rectangle with perimeter 36 is a 9 x 9 square—which gives him the idea that shapes with equal sides may give bigger areas and he starts to think about equilateral triangles. Niko is about to draw an equilateral triangle when he gets distracted by Jaden who tells him to forget triangles, he has found that the shape with the largest area made of 36 fences is a 36-sided shape. Jaden suggests to Niko that he find the area of a 36-sided shape too and he leans across the table excitedly, explaining how to do this. He explains that you divide the 36-sided shape into triangles and all of the triangles must have a one-meter base, Niko joins in saying, "Yes, and their angles must be 10 degrees!" Jaden says, "Yes, and to work it out we need tangent ratios which Lori has just explained to me."

Jaden and Niko move closer together, incorporating ideas from trigonometry, to calculate the area.

As the class progresses many students start using trigonometry, some students are shown the ideas by Lori, some by other students. The students are excited to learn about trig ratios as they enable them to go further in their investigations, they make sense to them in the context of a real problem, and the methods are useful to them. In later activities the students revisit their knowledge of trigonometry and use them to solve other problems.

Opportunities for learning – California Mathematics Standards

G-SRT.B.4: Prove theorems about triangles.

G-SRT.B.5: Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

G-CO.D.12: Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).

G-CO.D.13: Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

G-SRT.B.5: Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

G-SRT.C.6: Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

G-SRT.C.8: Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

G-MG.A.3: Apply geometric methods to solve design problems

G-SRT.C.8.1: Derive and use trig ratios for special triangles

NCTM’s (2014) *Principles to Action* offers further guidance on what teachers and students are doing in classrooms that focus on mathematics learning[1]:

NCTM, Principles to Action, 2014

Establish Mathematics Goals to Focus Learning: Teacher and Student Actions

What are *teachers* doing?	What are *students* doing?
Establishing clear goals that articulate the mathematics that students are learning as a result of instruction in a lesson, over a series of lessons, or throughout a unit. Identifying how the goals fit within a mathematics learning progression. Discussing and referring to the mathematical purpose and goal of a lesson during instruction to ensure that students understand how the current work contributes to their learning. Using the mathematics goals to guide lesson planning and reflection and to make in-the-moment decisions during instruction.	Engaging in discussions of the mathematical purpose and goals related to their current work in the mathematics classroom (e.g., What are we learning? Why are we learning it?) Using the learning goals to stay focused on their progress in improving their understanding of mathematics content and proficiency in using mathematical practices. Connecting their current work with the mathematics that they studied previously and seeing where the mathematics is going. Assessing and monitoring their own understanding and progress toward the mathematics learning goals.

Use Open, Engaging Tasks

What are Open Tasks?

Open tasks are those that enable students to take ideas to different levels (Vale et al., 2012). When tasks have a low floor and a high ceiling, it means that any student can

[1] A link to the Fences task can be found in Youcubed, n.d.a.

access the task but the task extends to high levels (Boaler, 2016; Krainer, 1993). When questions are narrow and focused, only some students are cognitively challenged at an appropriate level, and the questions are often not very interesting. When tasks are open, they allow all students to work at levels that are appropriately challenging for them, within the content in their grade. Smith, et al. (2000)'s math task analysis framework offers helpful descriptions of narrow tasks, which they refer to as memorization and procedures without connections tasks, and open tasks, which they refer to as procedures with connections and doing mathematics tasks. Two examples of such tasks are given below; one based in a real-world context, and one that encourages exploration of mathematical ideas through numerical patterns. Both types of tasks should be offered.

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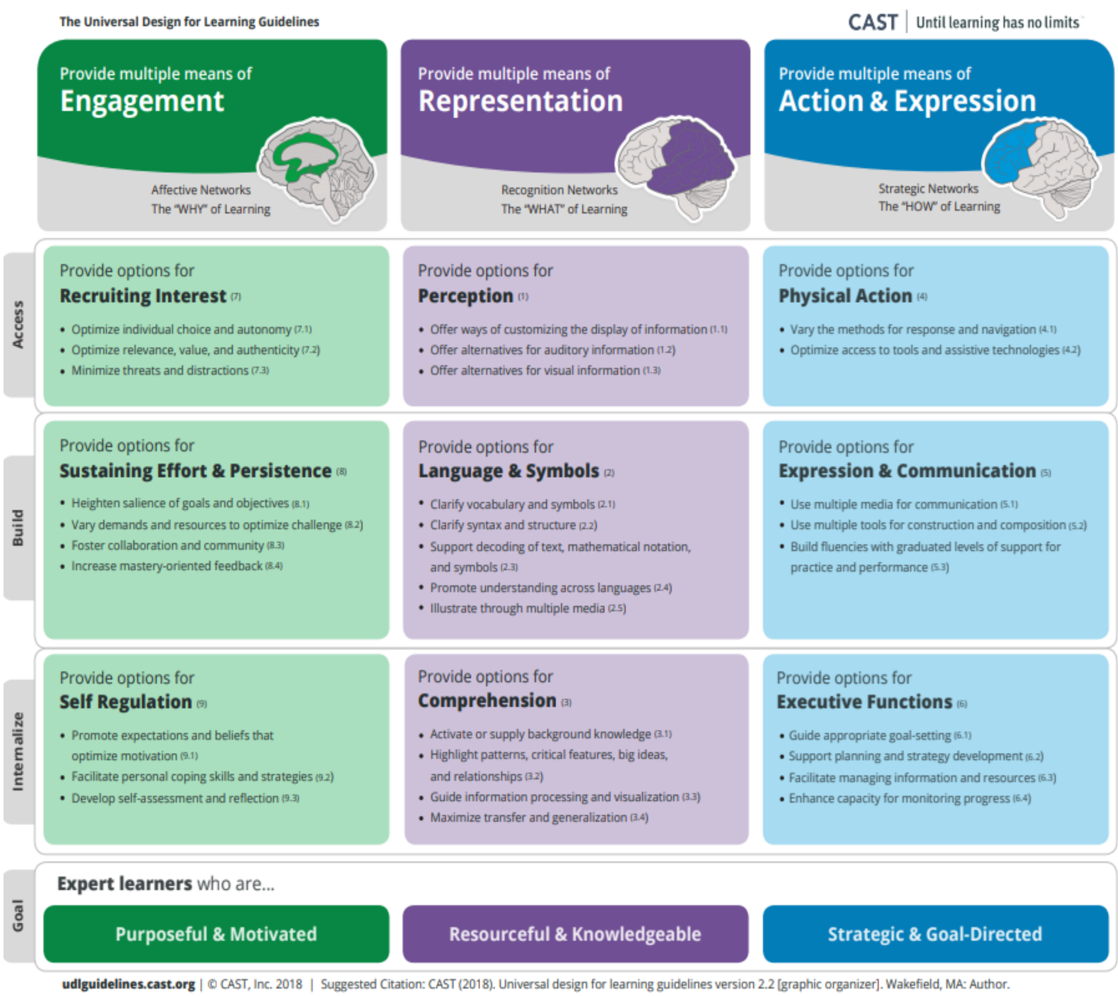
Lower-Level Demands	Higher-Level Demands
Memorization • involve either reproducing previously learned facts, rules, formulae or definitions OR committing facts, rules, formulae or definitions to memory. • cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure. • are not ambiguous. Such tasks involve exact reproduction of previously-seen material and what is to be reproduced is clearly and directly stated. • have no connection to the concepts or meaning that underlie the facts, rules, formulae or definitions being learned or reproduced.	Procedures with Connections • focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas. • suggest pathways to follow (explicitly or implicitly) that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts. • usually are represented in multiple ways (e.g., visual diagrams, manipulatives, symbols, problem situations). Making connections among multiple representations helps to develop meaning. • require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with the conceptual ideas that underlie the procedures in order to successfully complete the task and develop understanding.

Lower-Level Demands	Higher-Level Demands
Procedures Without Connection • are algorithmic. Use of the procedure is either specifically called for or its use is evident based on prior instructions, experience, or placement of the task. • require limited cognitive demand for successful completion. There is little ambiguity about what needs to be done and how to do it. • have no connection to the concepts or meaning that underlie the procedure being used. • are focused on producing correct answers rather than developing mathematical understanding. • require no explanations or explanations that focuses solely on describing the procedure that was used.	Doing Mathematics • require complex and non-algorithmic thinking (i.e., there is not a predictable, well-rehearsed approach or pathway explicitly suggested by the task, task instructions, or a work-out example). • require students to explore and understand the nature of mathematical concepts, processes, or relationships. • demand self-monitoring or self-regulation of one's own cognitive processes. • require students to access relevant knowledge and experiences and make appropriate use of them in working through the task. • require students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions. • require considerable cognitive effort and may involve some level of anxiety for the student due to the unpredictable nature of the solution process required.

Stein, Smith, Henningsen, and Silver, 2000, 16

Why Use Open Tasks?

Open tasks invite students to engage in multi-dimensional ways (Vale et al., 2012). They provide an ideal opportunity to integrate the important principles of UDL which recognize the strengths of learner variability and student diversity (CAST, 2018). Open tasks also connect to Darling's framework, in particular the suggestion to take an asset approach, include group work, and make work visual.



Long description of Universal Design for learning framework is available at https://udlguidelines.cast.org.

When students work on open tasks they take advantage of opportunities to engage in different ways, using multiple ways of representing mathematical ideas, and expressing understanding (see also Lambert, 2020). Open tasks provide teachers with opportunity to listen carefully, to make sense of student thinking, and to assess formatively as the lesson progresses. This creates learning opportunities that meet students where they are in their learning, supporting foundational knowledge-building for students grappling with core concepts, as well as opportunities for more advanced learners to make new mathematical connections. See Chapter 11 for further discussion of how the use of open tasks enables teachers to gather important information about students' learning.

Teachers must find out about and bring into mathematics the culture of their students, engaging in culturally relevant pedagogy (Aguirre, 2012; Ladson-Billings, 2009; Hammond, 2020). Listening to the questions that students wonder about can provide opportunities to design learning experiences around their mathematical curiosities. When teachers pay attention to the data standards in their grades, they can choose rich, open tasks to teach them, leading to data literacy in the early grades and, ultimately, an understanding of the new discipline known as "data science." Data science in school means learning to ask statistical investigative questions, collect, consider and analyze data and communicate findings. (See also Franklin and Bargagliotti, 2020.) Most Data Science[2] tasks, such as those highlighted in Chapter 5 (Data Science) and in Chapter 7 (Mathematics: Investigating and Connecting, Grades 6–8), are naturally open, and provide many opportunities for students to connect mathematics to their lives. Students can, for example, design wheelchair ramps, plan a new school garden, or survey peers to find out how they have been impacted by distance learning, drawing from their own knowledge and interests as they learn new mathematics. These tasks that draw from students' lives are very different from the imagined contexts that often fill textbooks and present mathematics in ways that students may find irrelevant and "other worldly" (Boaler, 2016). With carefully chosen projects students can learn to address the inequities they experience, learning mathematical tools that allow them to highlight inequities and plan new ways forward

[2] Data science is an interdisciplinary field that uses quantitative and representational strategies to extract knowledge and insights from noisy, structured and unstructured data, and apply knowledge and actionable insights from data across a broad range of application domains.

(see also Component 3, Teach Toward Social Justice, below, and Gutstein [2003, 2006]; Berry et al. [2020]).

Neuroscience research has shown that the most effective people have more active brain connections between different brain pathways (Menon, 2015; Kalb, 2017). Students encounter opportunities to develop brain connections when they see and experience mathematics in different ways. In one example, Park and Brannon (2013) found that when students worked with numbers and also saw the numbers as visual objects, brain communication was enhanced and student achievement increased. A range of different research studies throughout K–12 have shown the importance of visual thinking in mathematics (West, 2004; Alibali and Nathan, 2012; Boaler et al., 2016; Boaler, 2019). Researchers even found that after four 15-minute sessions of playing a game with a number line, differences in knowledge between students from low-income backgrounds and those from middle-income backgrounds were eliminated (Siegler and Ramani, 2008).

All mathematical ideas can be considered in different ways—visually, through touch or movement, through building, modeling, writing and words, through apps, games and other digital interfaces, as well as through numbers and algorithms. Fingers have been shown to be particularly important as a visual and physical representation for students, enabling the development of important brain areas (Boaler et al., 2016). The tasks used in classrooms should encourage multi-dimensional forms of engagement. Tasks that offer multiple ways to engage with and represent mathematical ideas also support students with identified learning differences (Lambert and Sugita, 2016), as well as students seeking greater challenges (Freiman, 2018). The UDL guidelines can support students with identified learning differences because they are designed to support learning for all (CAST, 2018).

Open Tasks to Meet a Wide Range of Needs

When math instruction is designed to offer open tasks, students can engage with the mathematics through many different pathways and tools with classroom discussions that are enhanced by the range of strategies and perspectives that students offer. For example, students benefit from discussing connections between direct modeling and more abstract reasoning strategies, helping students who may previously have relied on one strategy. Discussing those connections enriches students across a range of knowledge – students using direct modeling approaches might start to notice connections to more abstract ideas, helping them build understanding. Similarly, students utilizing more abstract strategies benefit from conceptually connecting those ideas to more concrete representations, drawings, or even other abstract approaches. By focusing on inclusive approaches to teaching, progress, not perfection, is the goal for each student. This focus supports differentiated learning in the sense that progress is built upon students' current understandings, allowing them to address any previously-unfinished learning even as they advance their thinking in powerful ways. One such type of approach is aligned with the principles of UDL, a framework for inclusive teaching. The principles, and their associated guidelines, are presented at the end of this chapter.

Rich classroom discussions at both the whole-class and small-group levels rely on the different strategies students bring and the ensuing approaches they take to articulate their thinking. When grappling with multidimensional tasks, collaborative student groups will also need ongoing support with participation structures in order to mitigate the status issues that can interfere with equitable participation. These participation supports will be discussed in the next section, Supporting Student Partnerships and Small-Group Work. This approach of providing open tasks while supporting participation not only supports learning, it serves to position students across a range of backgrounds as mathematical thinkers. Open, multi-dimensional tasks offer authentic opportunities for all students to contribute their unique perspectives. This start can engage all students and draw them into mathematical conversations on an equal footing. When students begin appropriately-structured group work or engage in classroom discussions of a mathematics problem or situation, they feel supported by an activity designed to use the different ways they see or think about the issue. And with appropriate language development scaffolds and strategies, all students can contribute in shared participation. Group work and mathematical discussions are productive when students share the intellectual work (Boaler, 2019b; Langer-Osuna, 2016; Langer-Osuna et al., 2020).

NCTM's (2014) *Principles to Action* offers guidance on how to select tasks for mathematical reasoning:

NCTM, Principles to Action, 2014

Implement Tasks that Promote Reasoning and Problem Solving: Teacher and Student Actions

What are *teachers* doing?	What are *students* doing?
Motivating students' learning of mathematics through opportunities for exploring and solving problems that build on and extend their current mathematical understanding. Selecting tasks that provide multiple entry points through the use of varied tools and representations. Posing tasks on a regular basis that require a high level of cognitive demand. Supporting students in exploring tasks without taking over student thinking. Encouraging students to use varied approaches and strategies to make sense of and solve tasks.	Persevering in exploring and reasoning through tasks. Taking responsibility for making sense of tasks by drawing on and making connections with their prior understanding and ideas. Using tools and representations as needed to support their thinking and problem solving. Accepting and expecting that their classmates will use a variety of solution approaches and that they will discuss and justify their strategies to one another.

Teaching with Open Tasks

Open, multi-dimensional tasks invite student discussion (Stein and Smith, 2011). Students are likely to see open tasks in different ways, and therefore respond with different assets in discussing problem approaches and solutions. This diversity in approach and thinking is generative because students are given the opportunity to make sense of mathematical ideas from multiple perspectives, which supports conceptual understanding and strategic reasoning (National Research Council, 2001; Stein and Smith, 2011).

Classrooms that use open tasks and encourage mathematical reasoning often have a similar structure:

- The teacher launches a problem (or problem context) and participation structures to support equitable engagement (Featherstone et al., 2011).
- Students work through the problem in peer partnerships or small groups (provide individual think time before peer talk).
- The class gathers for whole-class discussion based on students' solutions and reflection (Smith and Stein, 2011).

Talk Moves

ADD ON
"I would like to add on to what … said."

REASONING
"I agree because…."
"I disagree because…."
"This is true because…."

REPEATING
"I heard you say…."
"Can you repeat what you said?"

REVOICING
"So, you are saying…."
"What I think you said was…."
"Did you mean…?"

SAY MORE
"Can you say more about that?"
"Can you give us more examples?"

PRESS FOR REASONING
"Why do you think that?"
"What is your evidence?"

Planning to teach in this way means educators must attend to the ways they can support—rather than control—student thinking. Smith and Stein's text, *5 Practices for Orchestrating Productive Mathematical Discussions* (2011), offers a useful approach to planning and implementing such tasks effectively. Chapin, O'Connor, and Anderson's book, *Classroom Discussions* (2013), also offers useful tools for facilitating productive classroom discourse. Useful classroom activities are also illustrated in the following vignettes about productive partnerships and peer re-voicing.

Anticipating the strategies students might use and the challenges and confusions students are likely to encounter allows the teacher to strategically plan questions before lessons begin (Smith and Stein, 2011). During planning, teachers should understand not only the needs of English learners, but also their myriad assets, such as their linguistical and cultural diversity, and design instruction that is universal and accessible to all. In the recommendations below, developed for the teaching of addition and reasoning (Lagunoff et al., 2015), the language specialists suggest paying careful attention to the terminology.

When students participate as audience members for classmates' presentations and explanations of the models and strategies that they used—when they observe others describing their reasoning—students ultimately determine whether or not the explanations clearly describe the learning and respond with ways the explanations could have been improved. Also, through limited prompting and strategic support from the teacher, students determine whether their peers have used correct terminology (e.g., add, subtract, one-digit, two-digit) when describing their processes. To support students at the **Emerging** level of English proficiency, the teacher provides more substantial support (Moschkovich, 2013). For example, she ensures that students understand the specific term under discussion (e.g., one-digit, two-digit) and asks a direct question such as, "Mary said this is a two-digit number" as she points to a number. "Is this a two-digit number?" (Lagunoff et al., 2015).

Teachers are encouraged to align instruction with the outcomes of the California ELD Standards, which state that linguistically and culturally diverse English learners receive instruction that values their home cultures. This instruction recognizes students' primary language as an *asset* and draws on them to build new learning (Moschkovich, 2013). It views language as a resource rather than a deficit, and treats students' every-day and home languages as linguistic resources to engage students in mathematics (Moschkovich, 2013). To do so, Moschkovich (1999) suggests that teachers listen for the mathematical ideas being expressed by students, noticing how students might draw on multiple language bases (i.e., translanguaging), extra-linguistic communication such as gesturing and using representation. Teachers could then re-voice students' ideas (for example, a teacher might say, "So I hear you say that this shape is not a triangle because it has four sides and triangles only have three sides. Is that right?") to both check their understanding of students' expressed ideas and to also offer the idea back to the student with potentially clearer mathematical language.

The work of anticipation starts with working through the day's task as part of planning, as well as thinking about one's individual students their mathematical strengths, in order to ensure that all students have access to the task. If teachers listen closely to students' thinking during the lesson by using classroom discourse as formative assessment (Cirillo and Langer-Osuna, 2018), they can make use of the questions they have prepared strategically and responsively, and support all students, including English learners, as they learn the content. This formative assessment—the in-the-moment work of teaching—is some of the most important (Munson, 2018). Despite careful pre-planning, however, surprises still arise during lessons. Teachers need to be flexible, improvising additional questions and prompts that might support emerging understanding and enable students to communicate the mathematics more coherently. To do this kind of work effectively, teachers need to feel supported when feeling vulnerable or uncertain; they need resources and time to do their best, then reflect on what worked and what did not in order determine next steps. Through this, they grow professionally as student-centered, responsive educators. Expert guidance from a mathematics coach or involvement in a long-term professional learning program can support teachers as they develop their capacity to leverage students' responses to maximize learning.

Classrooms should include, at times, the use of real-world data. These data should be rooted in contexts students can engage with as a way to understand mathematics as an important tool for participating meaningfully in their community. Mathematics is a quantitative lens through which to view the patterns that exist throughout the world. When grappling with the data, students can pose questions about issues that matter to them, drawing upon content from relevant issues like cyber bullying, neighborhood resources, sports and recreation, or water quality, among endless others. Data related to these and other issues can draw from not only a range of mathematical ideas and curiosities from students, but from a range of feelings about relevant, complex issues. This focus on complex feelings aligns with trauma-informed pedagogy, which highlights the importance of allowing students to identify and express their feelings as part of mathematics sense-making, and to allow students to address what they learn about their world by suggesting recommendations and taking action (Kokka, 2019). However, not all mathematics problems need to be related to the world—students can be fully engaged exploring pure number patterns, for example. In the following examples, two problems are highlighted; one is purely numerical, and one draws from real-world data.

Two Examples of Open Tasks

Example 1

Four 4s. How many numbers can you create that have values between 1 and 20 using exactly four 4s and any operation?

Source: (Youcubed, n.d.b.)

Opportunities for Mathematics Content Learning	Opportunities for Mathematics Practices Learning	Opportunities for Language Development and Teacher Actions
Order of operations: 3. NBT Number and operations in Base Ten • Use place value understanding and properties of operations to perform multi-digit arithmetic. 6. EE Expressions and Equations • Apply and extend previous understandings of arithmetic to algebraic expressions o Write and evaluate numerical expressions involving whole-number exponents.	MP 1) Make sense of problems & persevere in solving them MP 2) Reason abstractly and quantitatively MP 3) Construct viable arguments & critique the reasoning of others	ELD PI.A - allow time for struggle; ask: • How could you get started on this problem? • What does it mean that “any operation” is allowed? • What does this symbol (parentheses, equal sign, fraction bar) mean to you?

(continued) 3. NF Number and Operations – Fractions • Develop understanding of fractions as numbers 4. NF Number and Operations – Fractions • Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers. 5. NF Number and Operations – Fractions • Use equivalent fractions as a strategy to add and subtract fractions. • Apply and extend previous understandings of multiplication and division to multiply and divide fractions. 1. OA Operations and Algebraic Thinking • Addition and Subtraction 3. OA Operations and Algebraic Thinking • Represent and solve problems involving multiplication and division.	(continued)	(continued) • What math language will help you prove your answer? ELD P2.C - allow time for rehearsal of response; ask: • How are those two examples connected? • How could that be written as one equation? ELD P1.A (above) ELD P1.C - encourage practicing language with partner; ask: • Which words on the word wall could help express this? • How else could we explain this answer?

Opportunities for Mathematics Content Learning	Opportunities for Mathematics Practices Learning	Opportunities for Language Development and Teacher Actions
(continued) • Understand properties of multiplication and the relationship between multiplication and division. 8. EE Expressions and Equations • Work with radicals and integer exponents. 3. OA Operations and Algebraic Thinking Number sense • Solve problems involving the four operations, and identify and explain patterns in arithmetic.	(continued)	(continued)

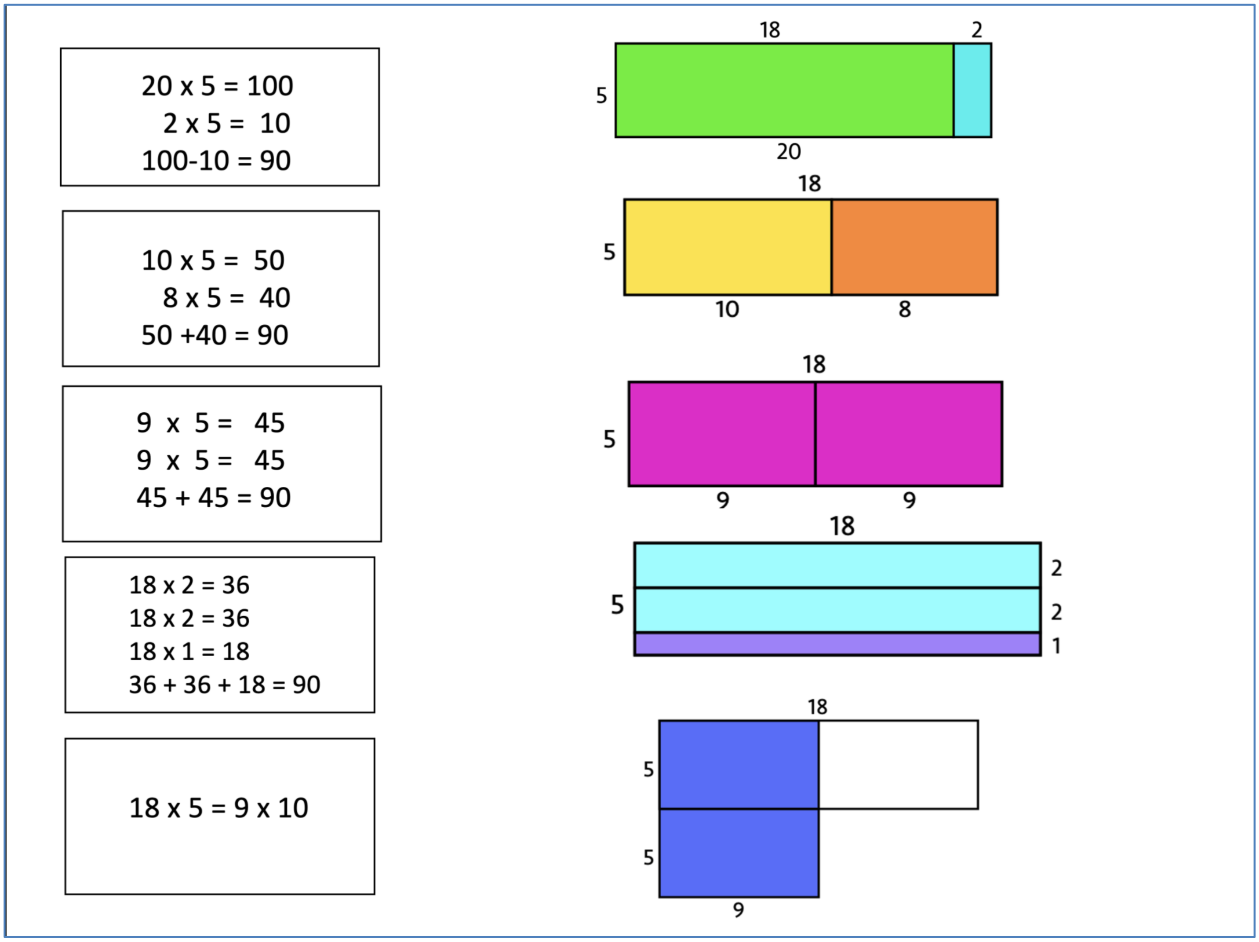
Example 2

Who attends your school? Which racial and gender groups are represented? And how does your school data compare to state or national data?

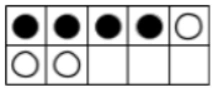
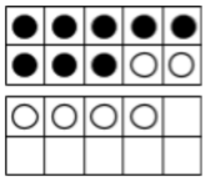
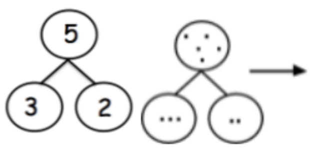
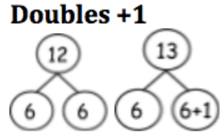
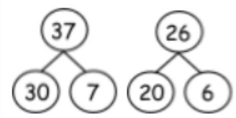
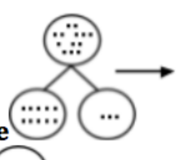
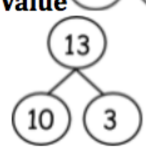
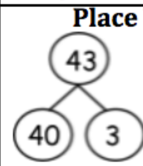
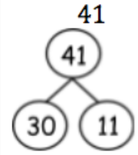
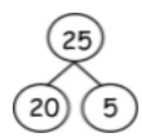
Opportunities for Mathematics Content Learning	Opportunities for Mathematics Practice Learning	Opportunities for Language Development and Teacher Actions
Construct a survey 7.SP Statistics and Probability Use random sampling to draw inferences about a population Measures of spread 6.SP Statistics and Probability Develop understanding of statistical variability Collect and analyze data 8.SP Statistics and Probability Investigate patterns of association in bivariate data Use a spreadsheet: SMP.5 Alg. II/ Mathematics III: Statistics and Probability: S-ID: Interpreting Categorical and Quantitative Data Ratios 6.RP Ratios and Proportional Relationships Understand ratio concepts and use ratio reasoning to solve problems.	1) Make sense of problems and persevere 4) Model with mathematics 5) Use appropriate tools strategically	ELD PI.A - allow time for struggle; ask: • How could you get started on this problem? • What does random sampling mean? • What does it mean to draw inferences? ELD P2.C - allow time for rehearsal of response; ask: • What inferences can you draw based on these patterns? patterns do you notice? • What inferences

Opportunities for Mathematics Content Learning	Opportunities for Mathematics Practice Learning	Opportunities for Language Development and Teacher Actions
(continued) Number sense 6.NS Number System Apply and extend previous understandings of numbers to the system of rational numbers	(continued)	(continued)

Number talks are a pedagogical practice that involve discussing numbers in ways that “open” these kinds of problems and expand the ways students encounter them (see also variations, such as graph talks (Marzocchi et al., 2019), data talks, and math movement (Vanderwerf, 2019). For example, a student can work on a question such as 18 x 5 in a textbook question, or in response to a teacher question, with the expectation that one answer is the goal. Alternatively, teachers can “open” the problem by presenting it as a number talk. In a number talk, teachers ask the class of students to work out the answer to 18 x 5, mentally and then share with the teacher when they have a solution, using a “quiet thumb.” Teachers then ask the class for the different answers that students may have found, and write them on the board. After the different answers are collected teachers can ask if anyone would like to explain their thinking. Ideally, different students will share different ways of thinking about the problem, with visual, as well as numerical solutions. For example:



Other visual examples are provided here (San Francisco Unified School District Mathematics Department. n.d.):

	Kindergarten	Grade 1	Grade 2
Ten Frame	 4 and 3 make 7 7 decomposes into 4 and 3 4 + 3 = 7 7 – 3 = 4 3 + 4 = 7 7 – 4 = 3		
	 8 and 6 make 14 A 10 and 4 more make 14 14 decomposes into 8 and 6 14 decomposes into 10 and 4 8 + 6 = 14 14 – 6 = 8 6 + 8 = 14 14 – 8 = 6		
Number Bond	 3 and 2 make 5 5 decomposes into 3 and 2 3 + 2 = 5 5 – 2 = 3 2 + 3 = 5 5 – 3 = 2	**Doubles** **Doubles +1**  6 + 6 = 12 6 + 7 = 13 6 + 6 + 1 = 13 12 - 6 = 6 13 – 6 = 7 13 – 7 = 6	**Decompose by Place Value** 26 + 37  20 + 30 = 50 6 + 7 = 13 50 + 13 = 63
	 **Place Value**  10 and 3 make 13 13 decomposes into 10 and 3 10 + 3 = 13 13 – 10 = 3 3 + 10 = 13 13 – 3 = 10	**Place Value**  40 and 3 make 43 43 decomposes into 40 and 3 40 + 3 = 43 43 – 40 = 3 3 + 40 = 43 43 – 3 = 40	41 - 25   11 – 5 = 6 30 – 20 = 10 6 + 10 = 16

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Number talks were created by Ruth Parker and Kathy Richardson, and have been developed in several books and video and online resources, given below. Any number problem can be used with students across K–12. When students become familiar with different mathematical strategies, visuals, and approaches, they feel more prepared to engage in open tasks.

Resources for Teaching “Number Talks:”

Humphreys, C & Parker, R. (2015) Making Number Talks Matter: Developing Mathematical Practices and Deepening Understanding, Grades 3–10. Stenhouse.

Humphreys, C & Parker, R. (2018) Digging Deeper: Making Number Talks Matter Even More, Grades 3–10. Stenhouse.

Parish, S. (2014). Number Talks: Helping Children Build Mental Math and Computation Strategies, Grades K–5, Updated with Common Core Connections. Math Solutions.

See the Math Talks section of Chapter 3 for further discussion of and resources for number talks.

Launching Open Tasks

To successfully launch tasks, teachers should discuss key contextual features and mathematical ideas, soliciting ideas from students to create shared language for anything that might be unfamiliar or confusing without reducing the cognitive demand of the task (see Lagunoff et al., example above). Whole-class discussions during the launch are also important opportunities to support students in learning how to effectively and inclusively share ideas during small group work. The vignette below describes an example of such a discussion in a fourth-grade classroom:

Vignette – Productive Partnerships

(Langer-Osuna, Trinkle, & Kwon, 2019)

Tracy, a fourth-grade teacher, joins her students at the carpet in the front of the room to launch the day’s lesson on place value. One of the first lessons of the year, she introduces the idea of “productive partnerships” with students before releasing them into small group work. When productive partnerships are the norm in a classroom, students engage in and strengthen their capacity for several mathematical practices, particularly SMPs 1, 3, 5, and 6, all of which involve reasoning, representing mathematical ideas, and communicating. She wants to use the informal nature of this portion of the lesson to illuminate how math “is organized in different text types and across disciplines using text structure, language features, and vocabulary depending on purpose and audience.” The students will make use of several mathematical practices (e.g., SMP.1, 2, 3, 6, 8), and will build skills as they invent and solve calculation problems using the four arithmetic operations (4.OA.4; 4.NBT 4, 5, 6). Tracy has planned her lesson carefully, making it accessible for her students by aligning her expectations with the principles of UDL, particularly encouraging students to represent their ideas in multiple ways—visually, numerically, and physically.

Tracy begins by asking students what it means to be productive. Students talk with a partner and offer different perspectives and ideas to the whole class. She then calls on a student volunteer to pretend to be her partner and act out what the class suggests they try to work “productively” as a partnership.

T: How can we show that we are ready to work with our partners?

S: Sit!

T: We should sit? Ok, let’s sit. How should we sit?

Students offer different ideas—sit facing each other, sit side-by-side to share the materials—which Tracy and her student partner model for the class. Tracy solicits suggestions for how they might attend to each other, decide on turns, or what to do if they reach a disagreement. After discussion, she tells the class that they will try out these ideas in their partnerships today, then moves on to launch the day’s mathematics problem: Four 4s. The four 4s task can be used at any grade level.

Tracy is confident that all her students will be able to engage in this open task utilizing their unique strengths. Her linguistically and culturally diverse students, especially the

English learners, will experience important learning opportunities through communicating their reasoning to their partners and contributing to the class discussion. Tracy relies on the CA ELD Standards which, in grade four, specify that Tracy's English learners will "develop an understanding of how language is a complex, dynamic, and social resource for making meaning." Tracy posts the problem statement on the whiteboard; she asks the students to read it silently first and then leads a choral reading: "Can we find every number between 1 and 20 using exactly four 4s and any operation?"

She signals for quiet thinking time, and after a few seconds, says, "When I first read this problem, I was not sure what it meant for us to do. Which words in this problem might have caused me confusion?" She uses a think aloud strategy, repeating, "BLANK confused me because…. BLANK confused me because…." After another pause, she asks the students to turn to a partner and ask, "What confused me?" The chatter provided formative feedback, and Tracy continues by prompting them to discuss what they think it means—which mathematical operations can they think of to use? "Try to be ready to explain what we should do, or perhaps share an example of a number you were able to find between 1 and 20 using exactly four 4s. In a few minutes, we will share our ideas with the whole class."

Partners turn toward each other to begin their discussion of the task. Partner discussions are based on an integrated ELD strategy called Three Reads constructive conversations (Los Angeles Unified School District, n.d.), where students first read to understand, then read to identify and understand the math, then read to make a plan. Their discussion is framed by cues on the board: "1.) Understand; 2.) Understand the math; 3.) Make a plan." She observes that many students are stuck between the second and third stage; they are not entirely sure of how to proceed, especially with regard to using all the operations. Many of the students have limited themselves to addition and are ready to suggest one way to get 16.

For example, one pair describes what they think the problem asks them to do:

Partner 1: Well, we can add all the fours together, and that makes 16.

Partner 2: Yeah, that works, but aren’t we supposed to get *all* the numbers from 1 to 20 as our answers? How are we supposed to do that?

Partner 1: Oh. What else can we do with the fours?

Tracy brings the class together to thank the students for their successful productive partnerships and to begin discussing what the problem asks and what solutions students have discovered.

Supporting Student Partnerships and Small-Group Work

Students can explore the mathematics inside open, multi-dimensional tasks in collaboration with peers. In order to realize the many benefits of student-led work, students must learn to share and discuss ideas inclusively. Issues of status, stereotypes, and peer relationships can get in the way of mathematical sense-making by biasing who participates and in what ways to the mathematical work at hand (Cohen and Lotan,1997; Esmonde and Langer-Osuna, 2011; Shah, 2017; LaMar, Leshin, and Boaler, 2020; Turner, Dominguez, Maldonado, and Empson, 2013). Established classroom norms and routines can support students in attending to and making sense of their peers’ mathematical ideas in ways that position one another’s thinking as worthy of taking into consideration (see also Cabana, Shreve, and Woodbury, 2014). Chapin, O’Connor, and Anderson (2013) provide further support for teachers in supporting productive classroom discussions, considering the mathematics to talk about, and the moves that encourage productive discussions.

Approaches described in this chapter can benefit all students, but they may be particularly useful for vulnerable students, including students learning English, students with identified learning differences, and students from racial and ethnic communities that have been historically marginalized in traditional math classrooms. A positive learning environment relies on foundational supports that are broadly available and incorporated into the classroom norms. Creating an inclusive mathematics classroom means incorporating strategies that support the participation of all students, with particular attention to under-served students. One approach to support participation of linguistically and culturally diverse English learners in mathematical discussions is outlined by Moschkovich (1999):

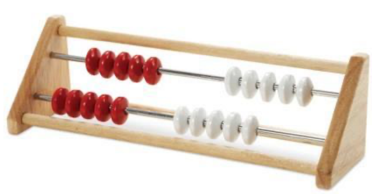
Teachers should attend to the mathematical ideas being expressed rather than focusing on correcting vocabulary. By instead revoicing and rephrasing students' statements, the teacher allows the student the right to evaluate the correctness of the teacher's interpretation. Second, revoicing helps keep the discussion mathematical by reformulating the statement in ways closer to the standard mathematics discourse.

The following vignette highlights a particular routine—peer revoicing—that helped first graders take turns sharing, listening, and making sense of one another's math ideas.

Vignette – Peer Revoicing

(Langer-Osuna, Trinkle, and Kwon, 2019)

Hope, a grade one teacher, introduces peer revoicing during a whole-class carpet discussion. She wants her young learners to practice a way of interacting that supports mutual attention and making sense of one another's mathematical thinking (SMP.3, 5, 6). Using a large rekenrek, she models revoicing with a student partner. The student partner first said how many beads she sees on the Rekenrek and how she knows (DI 1, CC 2; 1.OA.3, 6).



S: I see eight beads because there are five on the top and three on the bottom and that's five, six, seven, eight.

T: So, I hear you say that you see eight beads because there are five beads on the top and three beads on the bottom and you counted up from five, six, seven, eight. and that's how you knew there were eight. Is that right?

S: [nods head] Yup.

Hope then modeled the language used for the revoicing. “Let’s practice that” she said to her class. “I hear you say ‘mmmmm,’ is that right?”

The class repeated as a chorus, “I hear you say ‘mmmmm,’ is that right?”

Students then practiced at the carpet with their partners, drawing on sentence frames taped onto the wall as needed, and a class set of rekenreks before taking their rekenreks back to their tables for partner work.

At their table, students took turns representing numbers. Ana represented the number 10 and turned it toward her partner Sam. Sam counted the beads one by one and then stated:

Sam: “I see a 10 because there are 1, 2, 3, 4, 5 on the top and 5 on the bottom.”

Ana: “So I hear you say, wait. Can you repeat?”

Sam: [giggles] I said I see a 10 because there are 5 on the top and 5 on the bottom and that makes 10.

Ana: “So I hear you saying that you see a 10 because there are 5 on the top and 5 on the bottom, is that right?”

Sam: “and that makes 10”

Ana: “and that makes 10. Is that right?”

Sam: Yes

Ana: Ok, my turn. You do a number now.

Revoicing is a talk move between two people where the contribution of the speaker is restated by the listener, who checks with the speaker to confirm understanding. It often includes a statement such as, “So I hear you say…” followed by a restatement of the speaker’s words and then a check for understanding, such as “Is that right?” Peer revoicing is a powerful routine for promoting both shared understanding of mathematics and mutual recognition as young mathematicians. Peer revoicing structures the dialogue between the speaker and the listener, ensuring that the contributions build meaningfully upon each other. Teachers can also intervene on status issues as they confer with groups of students. Complex instruction offers a status intervention technique where teachers strategically find opportunities to elevate the mathematical contributions of a student perceived as low-status by pointing out the student’s idea, strategy, or drawing as useful to the group and worthy of consideration by peers (Cohen and Lotan, 1997; Cabana, Shreve, and Woodbury, 2014; LaMar, Leshin, and Boaler, 2020).

Orchestrating Reflective Whole-Class Discussions

Whole-class discussions are good opportunities for teachers to listen closely to and facilitate students’ ideas as students work to articulate their thinking. To implement the SMPs, it is necessary that teachers give careful attention to the types of questions they use; high quality, probing questions empower students to deepen their understanding. But all too commonly, questions that demand only simple recall or superficial explanation dominate classroom conversation (Simpson et. al., 2014).

The Mathematics Assessment Project (MAP) offers a series of professional development modules (Mathematics Assessment Project, n.d.). One of these modules, *Improving Learning through Questioning,* includes guidance on how and why to use open-ended questions, and provides examples such as, “What patterns can you see in this data? or Which method might be best to use here? Why?” Questions of this type take students beyond simple recall of known facts; instead, they call for original thought and connections of concepts. MAP research has found that in order to be effective, questions must be designed to include all students and to elicit thinking and reasoning. Teachers should provide think time, avoid judging student responses, and pose follow-up questions that encourage continued thinking.

Whole-class discussions at the close of a lesson provide additional opportunities to reflect on the impact of student partnerships and small-group work so that students increasingly internalize the expectations and learn the tools of inclusive, productive, shared mathematical work. Teachers might ask, “What went well in your partnerships today that we can learn from? What was difficult? What might we try tomorrow to be better partners?” Responses not only allow students an opportunity to express their thoughts like a mathematician, but the responses can provide invaluable formative feedback for teachers to use when defining the next steps in the learning progression(s). NCTM’s (2014) *Principles to Action* offers further guidance on facilitating meaningful mathematics discourse:

NCTM, Principles to Action, 2014

Facilitate Meaningful Mathematical Discourse: Teacher and Student Actions

What are *teachers* doing?	What are *students* doing?
Engaging students in purposeful sharing of mathematical ideas, reasoning, and approaches, using varied representations. Selecting and sequencing student approaches and solution strategies for whole-class analysis and discussion. Facilitating discourse among students by positioning them as authors of ideas, who explain and defend their approaches. Ensuring progress toward mathematical goals by making explicit connections to student approaches and reasoning.	Presenting and explaining ideas, reasoning, and representations to one another in pair, small-group, and whole-class discourse. Listening carefully to and critiquing the reasoning of peers, using examples to support or counterexamples to refute arguments. Seeking to understand the approaches used by peers by asking clarifying questions, trying out others’ strategies, and describing the approaches used by others. Identifying how different approaches to solving a task are the same and how they are different.

Teach Toward Social Justice

Mathematics is a tool that can be used to both understand and impact the world. Mathematics has often been experienced by students as a subject area that is disconnected from everyday life; this has occluded possibilities for students to develop more personal and powerful relationships to mathematics, and has led too many students to believe mathematics is not for them. A different perspective enables teachers to not only help their students see themselves inside mathematics but develop knowledge and understanding that allows them to use mathematics toward betterment in their worlds.

Teaching toward social justice involves two connected aspects: First, teaching in ways that acknowledge students' cultural backgrounds, histories, and funds of knowledge (Culturally responsive teaching; see Hammond, 2020), so that students see mathematics as a set of lenses on the world that is relevant to their own lives. Second, incorporating into mathematics instruction students' authentic questions about social issues that mathematical tools—such as those involved in data-based investigations—can help to answer and impact.

Teachers can take a justice-oriented perspective at any grade level, K–12, helping students feel a sense of belonging (Brady et al., 2020), and empowering them with tools to address important issues in their lives and communities. In a special issue (TODOS, n.d.), TODOS, an affiliate organization of the National Council of Teachers of Mathematics, presents six articles written by educators who are involved in teaching and learning mathematics from a social justice perspective. Each author describes "the promises, tensions, and struggles of engaging themselves and others...in fundamentally changing the experience of learning and teaching mathematics." For example, Chao and Jones (2016) state, "Finally, mathematics for young children must involve play in order to open up opportunities for non-routine problem solving, practicing perseverance, and connecting mathematical ideas (Parks, 2015; Wager, 2013). Therefore, we situate social justice mathematics at the prekindergarten level as developing powerful mathematical identities, developing critical mathematics agency, honoring and connecting to children's family and cultural histories, and centered around play." (17). As one example of such an opportunity to explore mathematical ideas at the prekindergarten level in a social justice context, Chao and Jones (2016) describe students counting and comparing in order to express unfairness in a skit about Rosa Parks and the Montgomery bus boycott.

Culturally responsive teaching is inextricably linked with multicultural education; both approaches work toward equitable and humanizing high-quality education for all students. Indeed, the two terms are often used interchangeably. A framework for addressing culturally responsive teaching that gathered information from fifty states outlines eight competencies:

- Reflect on one’s cultural lens
- Recognize and redress bias in the system
- Draw on students’ culture to shape curriculum and instruction
- Bring real-world issues into the classroom
- Model high expectations for all students
- Promote respect for student differences
- Collaborate with families and the local community
- Communicate in linguistically and culturally responsive ways

Source: (Muñiz, 2019, 12).

Students come from many different backgrounds, experiences, and cultural identities. Culturally responsive teaching draws on students’ experiences through their family, community, and cultural and linguistic forms of knowing (Gonzalez, Moll, and Amanti, 2006), in ways that go far beyond food, music, and folklore. Such an approach is foundational to participating in the global economy.

Culturally responsive teaching can be implemented in mathematics by exploring students’ lives and histories, centering contributions that historically marginalized people have made to mathematics in the design and implementation of curricula, creating opportunities for teachers and students to share their own autobiographies as mathematics doers and learners, and creating spaces for students to participate as authors of their mathematical learning experiences.

Esmonde and Caswell (2010) give an example of a fifth-grade project in which mathematics (including data science) helps students explore questions of justice: Focused on access to water as a human right, the project integrates topics of volume, capacity, operations, and proportional reasoning to explore their families' usage of water and access to water in developing countries. As an example of culturally responsive teaching at the primary level, Esmonde and Caswell (2010) describe the Number Book Project, in which kindergarteners and their families share number stories, songs, and games that parents or others knew as children. They then design classroom activities that draw on these number stories, songs, or games.

In the example that follows (from Diez-Palomar and Lopez Leiva, 2018), a group of students explored their family's immigration experiences through a measurement lesson on the topic of unit conversion, specifically between the US system and the metric system. Many of the students had experienced immigrating with their families to the US or knew relatives who had, as well as had family members living in other countries. Through map explorations and a series of discussions, students used and expanded their math skills, as you see in the vignette below:

Vignette: Exploring measurements and family stories

(from Diez-Palomar and Lopez Leiva, 2018, 49)

On a map, [two] students located the different places where their relatives lived or that they had heard mentioned. They selected the starting and ending points of immigration and figured out the distances. The discussion continued:

Mary Jo: Yeah so right here to here. Like right here to right here is a mile.

Jocelyn: I think it's more than a mile.

Mary Jo: Eight miles?

Jocelyn: There's a scale on the map somewhere, let's look. Let's measure this, how long is this? Okay, first of all what are these numbers here, what do those represent?

Mary Jo: Inches, one inch.

Jocelyn: Then what are these numbers?

Mary Jo: Millimeters.

Jocelyn: What’s millimeters?

Mary Jo: Millimeters are more than, no.

Jocelyn: Do you see them mm? Where’s the mm?

Mary Jo: Oh, these are millimeters, these are inches. …”

Multicultural children’s literature can also be used as contexts to connect learning mathematics with students’ cultural experiences (Esmonde and Caswell, 2010; Leonard, Moore, and Brooks, 2013). For example, in *The Great Migration: An American Story* (Lawrence and Myers 1995), young children explore quantity in terms of population shifts. In *First Day in Grapes* (Perez, 2002), a boy from a family of migrant workers uses his knowledge of mathematics to earn the respect of his peers. Drawing on *The Black Snowman* (Mendez, 1989), students can explore money problems through contexts linked to the African Diaspora. *One Grain of Rice* (Demi, 1997) offers students a context for exploring exponents and the importance of sharing food through the story of a peasant girl who tricks a king into giving her the royal storehouse’s entire supply of rice. *Multicultural Mathematics Materials* by Marina Krause (2000) also includes several games and activities that draw on Hopi and Navajo materials.

Empowering students with mathematics also includes sending the message that learning is always unfinished. An important message for students is the value of taking mathematical risks, such as making mathematical errors and confusions, public in order to make sense of them together as a classroom of learners. This mindset creates the conditions for students to develop a sense of ownership over their mathematical thinking, normalizes mathematical struggle as part of learning, and positions all learners as belonging to the discipline of mathematics. For example, in the vignette below, the teacher, Ms. Wong, offers students an open-ended math task, which she then implements in a way that positions all learners as belonging to their mathematics sense-making community (SMP.3, 7, 8.).

Vignette

This vignette comes from research based in a California high school committed to social justice and a mathematics classroom designed to foster positive mathematics identities (Gargroetzi, 2020). The following is a transcript from the lesson that occurred.

Function 1

Directions: Copy the table below onto your binder paper:

Input	3	5	1	2	11	-3
Output	13	21	5			

1. Describe in words the function that converts the input to the output.
2. What would the output be for the following inputs? Show your work.
    a. 2?
    b. 11?
    c. -3?

Ms. Wong: Raise your hand if you tried a rule that didn’t work.

Gabriel raised his hand and was called on, shared an idea he had that didn’t work out.

Ms. Wong: I love that you just shared something that you tried that didn’t work! Raise your hand if you found a rule that does work.

Scott is called on. He explains he put the numbers in order least to greatest, then noticed that from 1 to 3 the outputs went up by eight, and from 3 to 5 the outputs went up by eight. So, from 1 to 2 the outputs should go up by four. So, he says, the pattern is plus four. Ms. Wong documents Scott’s thinking on the board using arrows to show the “plus four.” She nearly bounces with excitement.

Bay shares the rule he found: Add the number to itself, then multiply by two, then add one. (He demonstrates with 3 and then 5.)

Maple: I have the same rule [as Bay] but in a different format. Y=4x+1. (She goes up to the white board to demonstrate her thinking and puts up multiple examples.) Ms. Wong asks about x and y. Maple explains that y is the output and x is the input.

Jill: Mine is the same as Maple's—times 4, add one.

Ms. Wong: I'm up here amazed! But also raise your hand if you are confused? Talk with your group—which rule makes the most sense to you?

Ok. Raise your hand if you liked Scott's rule (no hands). It's okay, sometimes genius is misunderstood! Bay's rule? (Bay's table all raises their hands, but nobody else), Maple and Jill's rule (everyone else raises their hands). Okay, since Maple and Jill's was the most popular—I'm not saying that means it's correct—let's use it to check. Ms. Wong began the discussion about this task by intentionally asking students to share if they tried a rule that "didn't work." In doing so, she sent the message that ideas were valued for reasons beyond being correct, that doing mathematics sometimes involved errors or confusion, and broadened possible ways for students to participate, lowering the risk of contributing to discussions. These moves positioned all students in the classroom as mathematical thinkers, learners, and community members (Gargroetzi, 2020).

Learning is not just a matter of gaining new knowledge—it is also about growth and identity development. As teachers introduce mathematics to students, they are helping them shape their sense of themselves as people who engage with numbers in the world (Langer-Osuna and Esmonde, 2017; Boaler and Greeno, 2000). Teaching mathematics through discussions and activities that broaden participation, lower the risks associated with contributing, and position students as thinkers and members of the classroom community, are powerful ways to support students in seeing themselves as young mathematicians. However, even within a classroom that utilizes these approaches, stereotypes are often at play (Joseph, Hailu, and Boston, 2017; Langer-Osuna, 2011; Milner and Laughter, 2015; Shah, 2017) and get in the way of creating robust, productive, and inclusive sense-making mathematics classroom communities (Shah, 2017). Indeed, teachers need to work consciously to counter racialized or gendered ideas about mathematics achievement (Larnell, Bullock, and Jett, 2016). Teachers can begin with awareness that leads to action and positive change. Teachers can support discussions that center mathematical reasoning rather than issues of status and bias by intentionally defining what it means to do and learn mathematics together in ways that include students' languages, experiences, and interests. One way in which they can do this is by emphasizing and welcoming students' families into classroom discussions (González, Moll, and Amanti, 2006; Turner and Celedón-Pattichis, 2011, Moschkovich, 2013).

Vignette

(from Turner et al., 2013)

In the vignette below, the teacher emphasizes the importance of communicating mathematical ideas, and attending and responding to the mathematical ideas of others across languages (DI 1, CC 3, SMP.3, 6; 4.OA.4, 5).

This vignette comes out of classroom research on the participation of linguistically and culturally diverse English learners in mathematical discussions (Turner et al., 2013). It documents an actual classroom experience. The teacher and students (grades 4–5) are discussing multiplicative relations using a paper-folding task where students folded a piece of paper to make 24 equal parts. Note how the teacher and class members engage with Ernesto's thinking about the mathematics in this task. Ernesto is an English learner; by focusing attention on his reasoning, the teacher is validating his status as a contributor to the mathematical discourse within the class.

Teacher: Ernesto, nos dices cómo lo hiciste? (Ernesto, would you tell us how you solved it?)

Ernesto: Lo doblé cinco veces, a la misma (I folded it five times, the same way—) [Stands up to come to the front of the room]

Teacher: [Hands Ernesto a piece of paper to show his folds] A ver, escúchenlo. (Let's see. Let's listen to him.)

Ernesto: Lo doblé. cinco veces, igual. Así. (I folded it five times, equally. Like this.) [Folds paper five times in the same direction, using an accordion-like fold] [Unfolds paper] Y me da seis partes. (And it gives me six parts.)

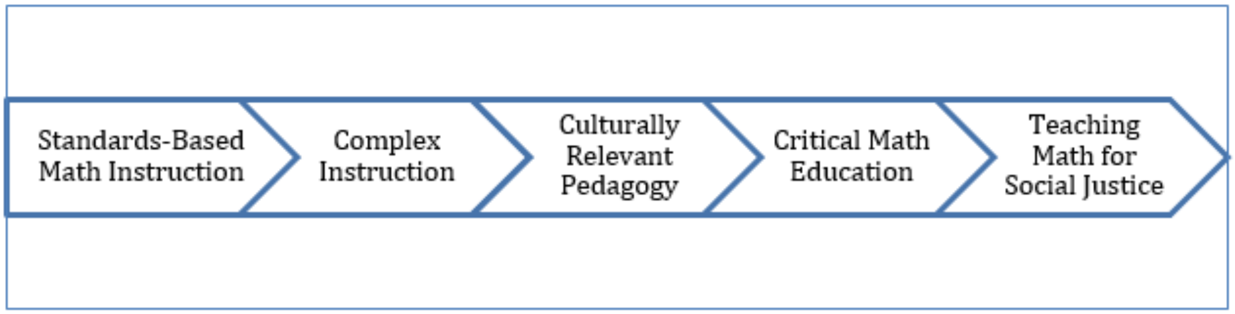
Teacher: His idea is to fold it five times, five times, and you get six parts. Does anyone have something to say to Ernesto? What do you think of how he did that? Anybody agree? [pause] Anybody else do it that way?

Corinne: It's different from ours, because he folded it five times to make six parts, and we—all three of us [the students who shared previously]—folded it in half, and [then] three times to make six parts.

Teacher: So, you noticed some way that Ernesto's strategy is a little bit different.

Reflection: The classroom community could be relied on to translate for others, and the emphasis remained on positioning all learners as thinkers and as members of the same community. In doing so, students who historically are marginalized from mathematical discussions—in this case, English Learners, were positioned as contributors and thinkers alongside their English-speaking peers. Further, students from dominant cultures—in this case monolingual English speakers—had the opportunity to engage with the mathematical ideas of typically silent students, to take their ideas into consideration, and to build on and make connections to their mathematical thinking.

Mathematics educators committed to social justice also work to both raise awareness of the ways textbook examples may exclude and stereotype certain students (Bright, 2016; Yeh and Otise, 2019) and to provide curricular examples that equip students with a tool kit and mindset to combat inequities with mathematics (Gutstein, 2006; Gutstein and Peterson, 2005; Moses and Cobb, 2001). The tasks have been developed to help students read and write the world with mathematics. First learning to use mathematics to highlight inequities—*reading the world* with mathematics—and then learning to change the world with mathematics—*writing the world* with mathematics (Gutstein, 2003; 2006). Note that these tasks correspond neatly to Drivers of Investigation (DI), DI 1(making sense of the world), DI 2 (predicting what could happen) and D3 (Impacting the future). Berry's approach builds upon four other bodies of work related to equitable teaching:



Source: Berry et al., 2020, 19.

A social justice approach to mathematics enables the *humanizing* of mathematics (Goffney and Gutiérrez, 2018; Su, 2020). Students start to see mathematics as something that relates to their lives and that can work to empower individuals and communities. In Ms. Wong's classroom, for example, tasks are not only deliberately designed to engage students in meaningful mathematics, but are also, at times, designed to support students in noticing that they are already important members of the mathematics classroom community, supporting belonging.

Vignette: Math Identity Rainbows

(from Wei and Gargroetzi, 2019)

Purpose: To reflect on and share the strengths that you and your teammates bring to the group

Each person will get six different colored strings. Each color represents a different math practice.

Your task is to arrange the cords according to your relative strengths and weaknesses.

Math Identity Rainbow Cords and Identification

- Pink is persevering: “I try my best and don’t give up, even when I face challenges.”
- Orange is numerical reasoning: “I have good number sense and use numbers flexibly.”
- Yellow is communicating: “I can explain my reasoning clearly to others.”
- Blue is modeling: “I can represent situations in everyday life mathematically to make predictions and solve problems.”
- Purple is pattern recognizing: “I can generalize patterns and see connections between concepts.”
- White is reflecting: “I know what I’ve learned and what I still need to learn.”

  Directions: Arrange the cords in the order of your strengths (strongest practices on top).

Through this task, Ms. Wong offered a definition of mathematical competence as multi-faceted. Ms. Wong emphasized, “All of these are extremely important to being mathematicians and everyone has these qualities but you have different strengths, right? So, the idea is you are going to order these cords on your desk so that the top strand is what you think your biggest strength is” (Gargroetzi, 2020). Students reflected individually and then shared their top strength with their partner. Students then discussed the strengths each group member brought to their mathematical work. In doing so, students had the opportunity to notice that together they were part of a mathematical community in which each member offered different, important strengths.

In the following vignette, a Ms. Ross leads students into a discussion of textbook questions, to consider the ways textbook examples may exclude and stereotype certain students (Bright, 2016; Yeh and Otise, 2019). As the students consider the questions, they learn the mathematics inside the questions and work to reformulate the questions to better reflect the students in the classroom (SMP.2, 3, 5, 6).

Vignette

Ms. Ross teaches fifth grade at the Jackie Robinson Academy. She has been focusing on developing her students’ sociopolitical consciousness through language arts and wants to bring mathematics into their thinking (SMP.1, 2). To begin the process, the class is led in an analysis of word problems from their fifth-grade mathematics textbook (NF.1, 2, 4, 5, 6). Ms. Ross selects three word problems to connect with the class's current read-aloud of *George*, a novel by Alex Gino that shares the story of a 10-year-old transgender fourth grader and her struggles with acceptance among friends and family. In doing so, the teacher is reflecting the recommendations of California's *Health Education Framework*, which suggests that sensitive discussions of gender are important for students (California Department of Education, 2021d). Ms. Ross reads the questions aloud to the class:

Amie used 7/9 yard of ribbon in her dress. Jasmine used 5/6 yard of ribbon in her dress. Which girl used more ribbon? How much more did she use?

A fifth grade class is made up of 12 boys and 24 girls. How many times as many girls as boys are in the class?

Ms. Hernandez knitted a scarf for her grandson. The scarf is 5/6 of one yard long and 2/9 of one yard wide. What is the area of the scarf?

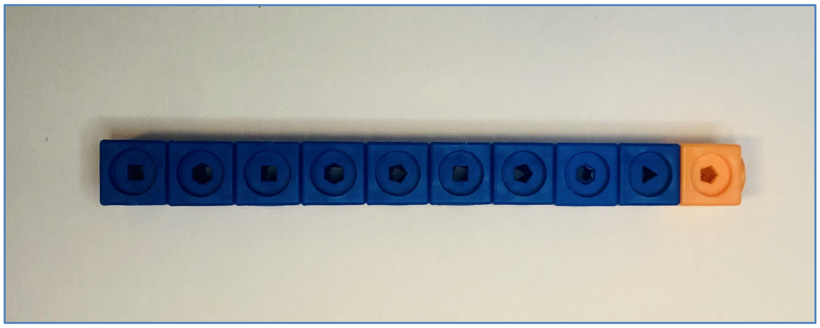
Ms. Ross uses a Say, Mean, Matter graphic organizer based on the following questions:

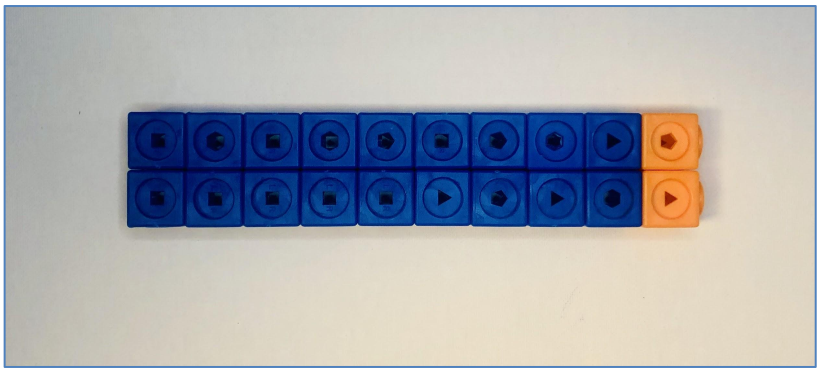
Say: What does the text say?

Mean: What does this mean? How do I integrate this? Read between the lines.

Matter: Why does this matter? Why does this matter to me or others? What are the implications?

Ms. Ross asked the first question, "What does this text say?" to engage her students in an analysis of the word problems with the question. Across the room, students sit forward in their chairs, some tracing the words on their papers with their fingers in attempts to examine the text more closely. After some silence, Ms. Ross repeats her question: "What does it say?" and students slowly begin to share out—explaining the word problems through words, drawings, and numbers—a process that transforms the class conversation into one of fractions and measurements.





Several students remark on the knitted scarf problem, leading to a group discussion comparing the characteristics and units of measure between area and volume. The excitement builds in the classroom as students traverse mathematical concepts, discussing different methods of problem solving based on what makes sense for each student and the methods of operation to use in relation to the problem context.

This vignette is adapted from: Yeh, C., & Otis, B. M. (2019). Mathematics for Whom: Reframing and Humanizing Mathematics. Occasional Paper Series, 2019 (41).

Invite Student Questions and Conjectures

Open tasks about big ideas in mathematics foster curiosity. Teachers can invite students to follow their curiosity by making space for their questions and conjectures. One of the most important yet neglected mathematical acts in classrooms is that of students asking or posing mathematical questions. These are not questions to help students move through a problem; they are questions that are sparked by wonder and intrigue (Duckworth, 2006). Examples of questions a student may ask include, “What is half of infinity?” “Is zero even or odd?” or “Does the pattern that describes the border of a square work if the shape is a pentagon?” Questions sparked by curiosity might also sound like pushing back on the ideas at play in the classroom, whether introduced by the teacher or peers. Students begin questions with, “But what about…?” or “But didn’t you just say…?” Such questions should be valued and students given time to explore them. They are important in the service of creating active, curious mathematical thinkers. Teachers can find more examples of good math questions in books by Peter Sullivan and Marion Small. For example, in Sullivan’s (2002) *Good Questions for Math Teaching*, he offers examples of good questions, organized by mathematical topics, that drive discussion, inquiry, and reasoning in math classrooms. Learning to ask good questions can be challenging for teachers; as teachers learn to engage in this practice, consider writing good questions down on a card and carry it around during class for reference (back pocket questions) or post them on the wall as a reminder until they become automatic.

Mathematically curious students who explore big ideas through open tasks are well primed to engage in another important mathematical act—that of making a conjecture. Most students in science classrooms know that a hypothesis is an idea that needs to be tested and proven. The mathematical equivalent of a hypothesis is a conjecture. When students are encouraged to come up with conjectures about mathematical ideas, and the conjectures are discussed and investigated by the class, students come to realize that mathematics is a subject that can be explored deeply and logically. It is through conjectures that curiosity and sense-making are nurtured. The Drivers of Investigation which are centered in this framework are intended to create opportunities for students to be curious and develop conjectures, as they work on investigations with the goal of “making sense of the world,” “predicting what could happen,” and/or “impacting the future.”

Snapshot

A teacher presented their fourth-grade students with a list of eight equations, noting that not all of them were true statements of equality. The students worked with partners to decide which were true, which were false, and to explain how they knew.

$2 \times (3 \times 4) = 8 \times 3$

$4 \times (10 + 2) = 40 + 2$

$5 \times 8 = 10 \times 4$

$6 \times 8 = 12 \times 4$

$9 + 6 = 10 + 5$

$9 – 6 = 10 – 5$

$9 \times 6 = 10 \times 5$

Ryan and Anen worked together, and after a few minutes, the teacher could see that they were very excited! The teacher stopped by their workplace and after listening to their explanation, and posing a few challenges, invited them to describe their “magic” trick with multiplication to the class. At the front of the class; Anen wrote equation c, $5 \times 8 = 10 \times 4$, on the board, and asked everyone to use a hand signal to show true or false. Almost all students indicated it is a true equation. Ryan asked the class about example d, $6 \times 8 = 12 \times 4$. Again, the class agreed that it is true.

Anen and Ryan continued, saying that something special was going on, and they had a conjecture they think *probably* works all the time, but they want to be sure. They explained that in $5 \times 8 = 10 \times 4$, they noticed “5” on the left side of the equation is half of the “10” on the right side, and the “8” on the left side is two times the “4” on the right side. So, they concluded, trying to use proper mathematical language, and pointing at the numbers as they spoke, “If you have factors like that where one first factor is half of the other first factor, and the second factor is twice as big as the other second factor, they’ll always be equal!”

The teacher called for the class to explore this conjecture and to see whether they could find a way to prove whether it is always true or not. Now the whole class was interested and trying to prove or disprove the Ryan/Anen conjecture.

The teacher supported the discussion in several ways by:

- bringing the class together to listen according to class norms such as, “everyone gets to speak” and “we listen carefully to each other’s ideas”
- encouraging the speakers to pause occasionally so that their classmates would have time to think and try out ideas
- asking students to repeat, revoice, or add on to each other’s statements
- re-stating Ryan’s and Anen’s explanations using precise mathematical terms
- checking with students who are learning English to ensure that they are both communicating with and supported by their partners during the student-led presentation
- calling for others in the class to express their own conjectures and challenges
- focusing students’ attention to Anen and Ryan’s explanations and questions
- posing questions to both the presenters and the other class members as the discussion progressed, such as:
    - why is this true?
    - will this always work?
    - does this work for other operations, or only for multiplication?
    - how can we know?
    - how are these numbers related?

Prioritize Reasoning and Justification

Reasoning is at the heart of doing and learning mathematics. Chapter 4 includes a description of how students come to conjecture, reason, and justify along with other important related acts together. All students can reason deeply with and about mathematics.

Open tasks invite students to reason about mathematics and, through discussion, justify their thinking. Reasoning with and about mathematics supports and enhances everyday life. A student who learns to reason about their ideas is learning to be a good communicator of mathematics, a skill that is essential in twenty-first century employment. Employers used to value highly the people who could calculate and come up with correct answers, but now computers perform calculations and employees are needed to program computers, make sense of solutions, reason about mathematical pathways and communicate their thinking so that other team members connect with them (Wolfram, 2020). Flexible and creative thinking is more highly valued in today's workplace than fast calculating (Mlodinow, 2018; Wolfram, 2020). NCTM's (2014) *Principles to Action* offers further guidance on supporting mathematical reasoning and justification among students:

NCTM, Principles to Action, 2014

Pose Purposeful Questions: Teacher and Student Actions

What are *teachers* doing?	What are *students* doing?
Advancing student understanding by asking questions that build on, but do not take over or funnel, student thinking. Making certain to ask questions that go beyond gathering information to probing thinking and requiring explanation and justification. Asking intentional questions that make the mathematics more visible and accessible for student examination and discussion. Allowing sufficient wait time so that more students can formulate and offer responses.	Expecting to be asked to explain, clarify, and elaborate on their thinking. Thinking carefully about how to present their responses to questions clearly, without rushing to respond quickly. Reflecting on and justifying their reasoning, not simply providing answers. Listening to, commenting on, and questioning the contributions of their classmates.

Reasoning is fostered when students have the opportunity to talk about mathematics with each other through whole class discussions and small group work on open tasks. Students can be given open tasks and asked to discuss ideas with each other and reason about them. This framework for teaching with open tasks is consistent with the recommendations from NCTM, in Catalyzing Change:

Mathematics Teaching Practices: Supporting Equitable Teaching Practices (NCTM, *Catalyzing Change in High School Mathematics*, 2018)

Mathematics Teaching Practices	Equitable Teaching
Establish mathematics goals to focus learning. Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals go guide instructional decisions.	• Establish learning progressions that build students' mathematical understanding, increase their confidence, and support their mathematical identities as doers of mathematics. • Establish high expectations to ensure that each and every student has the opportunity to meet the mathematical goals. • Establish classroom norms for participation that position each and every student as a competent mathematics thinker. • Establish classroom environments that promote learning mathematics as just, equitable, and inclusive.
Implement tasks that promote reasoning and problem solving. Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.	• Engage students in tasks that provide multiple pathways for success and that require reasoning, problem solving, and modeling, thus enhancing each students' mathematical identity and sense of agency. • Engage students in tasks that are culturally relevant. • Engage students in tasks that allow them to draw on their funds of knowledge (i.e., the resources that students bring to the classroom, including their home, cultural, and language experiences).

Mathematics Teaching Practices	Equitable Teaching
Use and connect mathematical representations. Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and to use as tools for problem solving.	• Use multiple representations so that students draw on multiple resources of knowledge to position them, as competent. • Use multiple representations to draw on knowledge and experience related to the resources that students bring to mathematics (culture, contexts, and experiences). • Use multiple representations to promote the creation and discussion of unique mathematical representations to position students as mathematically competent.
Facilitate meaningful mathematical discourse. Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing approaches to arguments.	• Use discourse to elicit students' ideas and strategies and create space for students to interact with peers to value multiple contributions and diminish hierarchal status among students (i.e., perceptions of differences in smartness and ability to participate). • Use discourse to attend to ways in which students position one another as capable or not capable of doing mathematics. • Make discourse an expected and natural part of mathematical thinking and reasoning, providing students with the space and confidence to ask questions that enhance their own mathematical learning. • Use discourse as a means to disrupt structures and language that marginalize students.

Mathematics Teaching Practices	Equitable Teaching
Pose purposeful questions. Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.	• Pose purposeful questions and then listen to and understand students' thinking to signal to students that their thinking is valued and makes sense. • Pose purposeful questions to assign competence to students. Verbally mark students' ideas as interesting or identify an important aspect of students' strategies to position them as competent. • Be mindful of the fact that the questions that a teacher asks a student and how the teacher follows up on the student's response can support the student's development of a positive mathematical identity and sense of agency as a thinker and doer of mathematics.
Build procedural fluency from conceptual understanding. Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.	• Connect conceptual understanding with procedural fluency to help students make sense of the mathematics and develop a positive disposition toward mathematics. • Connect conceptual understanding with procedural fluency to reduce mathematical anxiety and position students as the mathematical knowers and doers. • Connect conceptual understanding with procedural fluency to provide students with a wider range of options for entering a task and building mathematical meaning.

Mathematics Teaching Practices	Equitable Teaching
Support productive struggle in learning mathematics. Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.	• Allow time for students to engage with mathematical ideas to support perseverance and identify development. • Hold high expectations, while offering just enough support and scaffolding to facilitate student progress on challenging work, to communicate caring and confidence in students.
Elicit and use evidence of student thinking. Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.	• Elicit student thinking and make use of it during lesson to send positive messages about students' mathematical identities. • Make student thinking public, and then choose to elevate a student to a more prominent position in the discussion by identifying his or her idea as worth exploring, to cultivate a positive mathematical identity. • Promote a class culture in which mistakes and errors are viewed as important reasoning opportunities, to encourage a wider range of students to engage in mathematical discussions with their peers and the teacher.

Conclusion

This chapter has recommended five ways of organizing classrooms to encourage equitable outcomes and active student engagement: teaching big ideas, using open tasks, teaching for justice, supporting students' questions and conjectures, and prioritizing reasoning and justification. To encourage truly equitable and engaging mathematics classrooms teachers need to broaden perceptions of mathematics beyond methods and answers so that students come to view mathematics as a connected, multi-dimensional subject that is about sense making and reasoning, to which they can contribute and belong. To achieve this, teachers need to change classroom approaches from work on short questions to instruction that engages students in rich, deep tasks that honor students' ideas and thinking and draws on their backgrounds, interests, and experiences as resources. Several documents and frameworks have been referenced that offer ways to support linguistically and culturally diverse English learners and students with learning differences. In all cases these communicate principles of good teaching that can be extended to all students. The five components are supported by research and practice as ideas that will encourage a diverse group of students to see themselves as mathematically capable, with growth mindsets, and develop a curiosity and love of learning that will encourage them throughout their schooling. Note: the assessment practices that support this vision of mathematics teaching and learning, and the development of growth mindsets are shared in Chapter 12.

Long Descriptions of Graphics for Chapter 2

Figure 2.1. Grade 6 Map of Big Ideas

The graphic illustrates the connections and relationships of some sixth-grade mathematics concepts. Direct connections include:

- Variability in Data directly connects to: The Shape of Distributions, Relationships Between Variables
- The Shape of Distributions directly connects to: Relationships Between Variables, Variability in Data
- Fraction Relationships directly connects to: Patterns Inside Numbers, Generalizing with Multiple Representations, Model the World, Relationships Between Variables
- Patterns Inside Numbers directly connects to: Fraction Relationships, Generalizing with Multiple Representations, Model the World, Relationships Between Variables
- Generalizing with Multiple Representations directly connects to: Patterns Inside Numbers, Fraction Relationships, Model the World, Relationships Between Variables, Nets & Surface Area, Graphing Shapes

- Model the World directly connects to: Fraction Relationships, Relationships Between Variables, Patterns Inside Numbers, Generalizing with Multiple Representations, Graphing Shapes
- Graphing Shapes directly connects to: Model the World, Generalizing with Multiple Representations, Relationships Between Variables, Distance & Direction, Nets & Surface
- Nets & Surface directly connects to: Graphing Shapes, Generalizing with Multiple Representations, Distance & Direction
- Distance & Direction directly connects to: Graphing Shapes, Nets & Surface Area
- Relationships Between Variables directly connects to: Variability in Data, The Shape of Distributions, Fraction Relationships, Patterns Inside Numbers, Generalizing with Multiple Representations, Model the World, Graphing Shapes

Return to graphic