Mathematics Framework
Appendix A: High School Pathways

Second Field Review Draft

The Traditional High School Pathway

Most of us are familiar with the Algebra I–geometry–Algebra II sequence of high school mathematics courses, as it has been the most common pathway for decades. The six conceptual categories for the CA CCSSM at the high school level are Number and Quantity, Algebra, Functions, Modeling, Geometry, and Statistics and Probability. In the Traditional Pathway described in the CA CCSSM, the standards from these conceptual categories have been organized into the three courses of Algebra I, Geometry, and Algebra II. Despite having a new set of standards, as of 2013, the outline of the courses has not changed significantly, so the outlines below will look familiar to many. The standards for the Traditional Pathway, by course, begin on page 59 of the CA CCSSM (CDE, 2013).

Note that "Traditional Pathway" refers to the organization of content, not to teaching practices. Although these courses are traditional in their content, they should be taught through active student engagement, as set out in Chapters 2 and 8, and whenever possible students should see and work on content that is conceptually integrated.

Traditional Pathway Big Ideas

The state of California set out the most important mathematical content and practices by highlighting a collection of big ideas in mathematics, TK–10 in the Digital Learning and Standards Initiative (CDE, 2021). In this document, the CACCSSM content standards and Standards for Mathematical Practice in transitional kindergarten through grade ten were organized into a set of Big Ideas, which themselves are organized into the Content Connections.

Figure A.1 presents the progression of Big Ideas for the Algebra I and Geometry course sequence. The network maps, in Figures A.2 and A.4, highlight important and foundational content, shown as nodes, for each grade level. As students explore and investigate with the Big Ideas, they will likely encounter many different content standards and note the connections between them. The size of a node relates to the number of connections it has with other Big Ideas. The connections between Big Ideas are made when the two connected Big Ideas contain one or more of the same standards.

The colors in the nodes correspond to the Content Connections, Big Ideas, and Standards tables, Figures A.3 and A.5, which follow each of the network diagrams for the two courses. The Big Ideas (middle column) are situated within their broader Content Connection (left column), and the CACCSSM content standards (right column) which can be addressed for each Big Idea are indicated.

Figure A.1: A Progression Chart of Big Ideas through Algebra I and Geometry

Content Connections	**Big Ideas: Algebra I**	**Big Ideas: Geometry**
Communicating Stories with Data	Investigate data	Fairness in data
Communicating Stories with Data	Model with functions	Geospatial data
Communicating Stories with Data	n/a	Probability modeling
Exploring Changing Quantities	Function investigations	Trig explorations
Exploring Changing Quantities	Systems of equations	Triangle congruence
Exploring Changing Quantities	Features of functions	Triangle problems
Exploring Changing Quantities	n/a	Circle relationships
Exploring Changing Quantities	n/a	Points & slopes
Taking Wholes Apart, Putting Parts Together	Growth & decay	Triangle congruence
Taking Wholes Apart, Putting Parts Together	n/a	Transformations

Content Connections	Big Ideas: Algebra I	Big Ideas: Geometry
Discovering shape and space	Model with functions	Triangle congruence
Discovering shape and space	Investigate data	Transformations
Discovering shape and space	n/a	Circle relationships
Discovering shape and space	n/a	Geometric models

Figure A.2: High School Algebra I Big Ideas

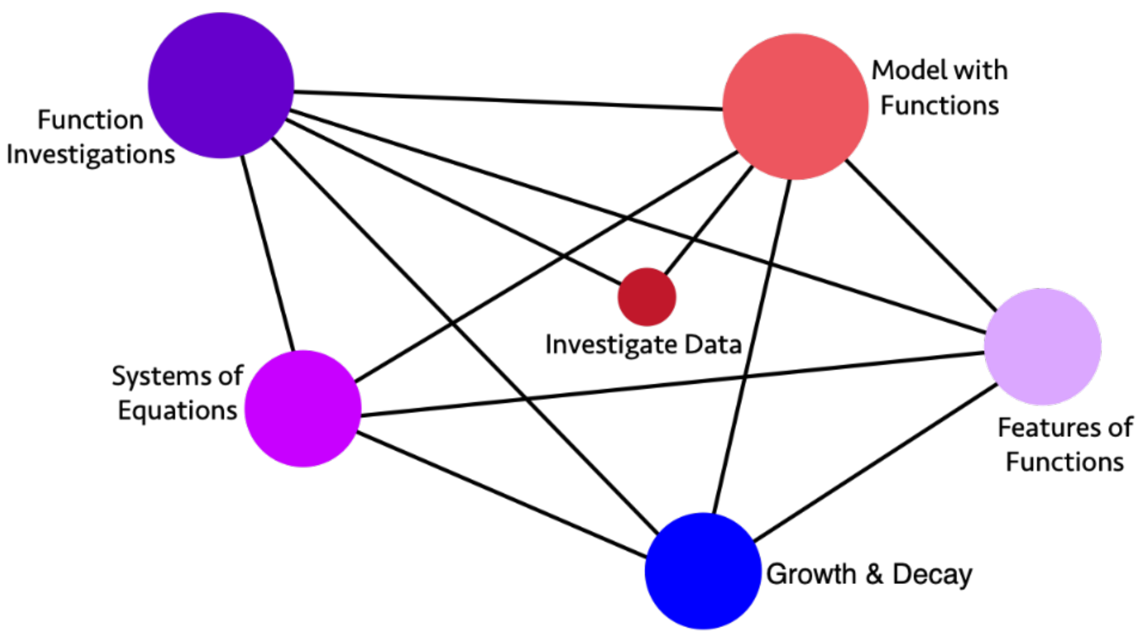

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Figure A.3: High School Algebra I Content Connections, Big Ideas, and Standards

Content Connection	Big Idea	Algebra Standards
Communicating Stories with Data & Discovering Shape and Space	**Investigate Data**	**S-ID.1, S-ID.2, S-ID.3, S-ID.6:** Represent data from two or more data sets with plots, dot plots, histograms, and box plots, comparing and analyzing the center and spread, using technology, and interpreting the results. Interpret and compare data distributions using center (median, mean) and spread (interquartile range, standard deviation) through the use of technology. ● Students have opportunities to explore and research a topic of interest and meaning to them, using the statistical methods, tools, and representations. ● Have students consider how different, competing interpretations can be made from different audiences, histories, and perspectives. ● Allow students to develop follow-up questions to investigate, spurred by the original data set.

Content Connection	Big Idea	Algebra Standards
Communicating Stories with Data & Discovering Shape and Space	**Model with Functions**	**F-IF.1, F-IF.2, F-IF.4, F-IF.5, F-IF.6, F-IF.7, F-IF.8, F-IF.9, F-BF.1, F-BF.2, F-BF.4, F-LE.1, F-LE.2, S-ID.5, S-ID.6, S-ID.7, S-ID.8, S-ID.9:** Investigate data sets by table and graph and using technology; fit and interpret functions** to model the data between two quantities. Interpret information from the functions, noticing key features* and symmetries. Develop understanding of the meaning of the function and how it represents the data that it is modeling; recognizing possible associations and trends in the data - including consideration of the correlation coefficients of linear models. ● Students can disaggregate data by different characteristics of interest (populations for example), and compare slopes to examine questions of fairness and bias among groups. ● Students have opportunities to consider how to communicate relevant concerns to stakeholders and/or community members. ● Students can identify both extreme values (true outliers) and data errors, and how the inclusion or exclusion of these observations may change the function that would most appropriately model the data. *intercepts, slope, increasing or decreasing, positive or negative ** functions include linear, quadratic and exponential
Exploring Changing Quantities	**Systems of Equations**	**A-REI.1, A-REI.3, A-REI.4, A-REI.5, A-REI.6, A-REI.7, A-REI.10, A-REI.11, A-REI.12, NQ.1, A-SEE.1, F-LE.1, F-LE.2:** Students investigate real situations that include data for which systems of 1 or 2 equations or inequalities are helpful, paying attention to units. Investigations include linear, quadratic, and absolute value. Students use technology tools strategically to find their solutions and approximate solutions, constructing viable arguments, interpreting the meaning of the results, and communicating them in multidimensional ways.

Content Connection	Big Idea	Algebra Standards
Exploring Changing Quantities	**Function investigations**	**F-IF.1, F-IF.2, F-IF.4, F-IF.5, F-IF.6, F-IF.7, F-IF.8, F-IF.9, F-BF.1, F-BF.2, F-BF.4, S-ID.5, S-ID.6, S-ID.7, S-ID.8, S-ID.9, F-LE.1, F-LE.2:** Students investigate data sets by table and graph and using technology; such as earthquake data in the region of the school; they fit and interpret functions to model the data between two quantities and consider the meaning of inverse relationships. Students interpret information from the functions, noticing key features* and symmetries. Students develop understanding of the meaning of the function and how it represents the data that it is modeling; they recognize possible associations and trends in the data - including consideration of the correlation coefficients of linear models. *one to one correspondence, intercepts, slope, increasing or decreasing, positive or negative
Exploring Changing Quantities	**Features of Functions**	**A-SSE.3, F-IF.3, F-IF.4, F-LE.1, F-LE.2, F-LE.6:** Students investigate changing situations that are modeled by quadratic and exponential forms of expressions and create equivalent expressions to reveal features* that help understand the meaning of the problem and situation being investigated. (driver of investigation 1, making sense of the world) Investigate patterns, such as the Fibonacci sequence and other mathematical patterns, that reveal recursive functions. *Factored form to reveal zeros of a quadratic function, standard form to reveal the y-intercept, vertex form to reveal a maximum or minimum.
Taking Wholes Apart, Putting Parts Together	**Growth & Decay**	**F-LE.1, F-LE.2, F-LE.3, F-LE.5, F-LE.6, F-BF.1, F-BF.2, F-BF.3, F-BF.4, F-IF.4, F-IF.5, F-IF.9, NQ.1, A-SEE.1:** Investigate situations that involve linear, quadratic, and exponential models, and use these models to solve problems. Recognize linear functions grow by equal differences over equal intervals; exponential functions grow by equal factors over equal intervals, and functions grow or decay by a percentage rate per unit interval. Interpret the inverse of functions, and model the inverse in graphs, tables, and equations.

Figure A.4: Big Ideas Map for Geometry

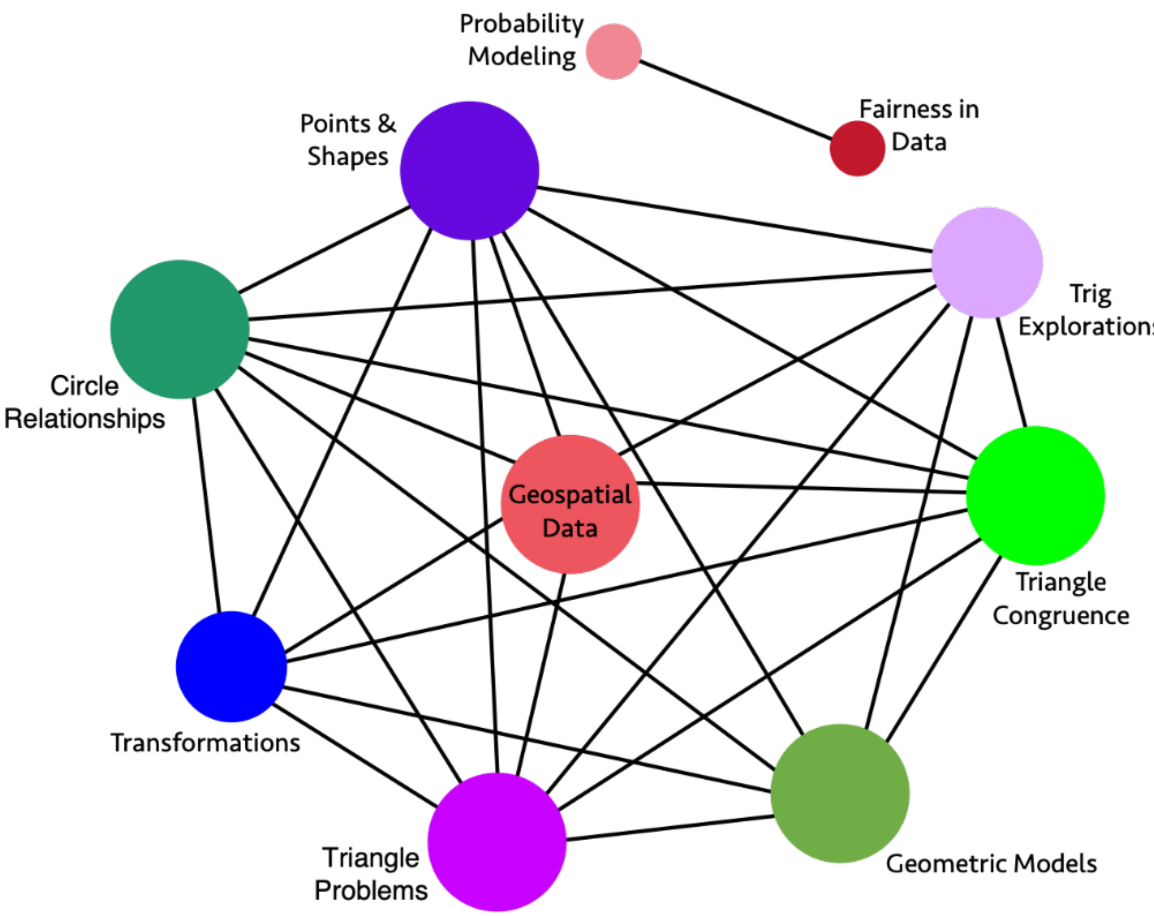

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Figure A.5: High School Geometry Content Connections, Big Ideas, and Standards

Content Connection	Big Idea	Geometry Standards
Communicating Stories with Data	**Probability Modeling**	**S-CP.1, S-CP.2, S-CP.3, S-CP.4, S-CP.5, S-IC.1, S-IC.2, S-IC.3, S-MD.6, S-MD.7:** Explore and compare independent and conditional probabilities, interpreting the output in terms of the model. Construct and interpret two-way frequency tables of data as a sample space to determine if the events are independent and use the data to approximate conditional probabilities. Examples of topics include product and medical testing, and player statistics in sports.

Content Connection	Big Idea	Geometry Standards
Communicating Stories with Data	**Fairness in Data**	**S-MD.6, S-MD.7:** Determine fairness and make decisions based on evaluation of outcomes. Allow students to explore fairness by researching topics of interest, analyzing data from two-way tables. Provide opportunities for students to make meaningful inference, and communicate their findings to community or other stakeholders.
Communicating Stories with Data	**Geospatial Data**	**G-MG.1, G-MG.2, G-MG.3, F-LE.6, G-GPE.4, G-GPE.6, G-SRT.5, G-CO.1, G-CO.2, G-CO.12, G-C.2, G-C.5:** Explore geospatial data that represent either locations (e.g., maps) or objects (e.g., patterns of people's faces, road objects for driverless cars), and connect to geometric equations and properties of common shapes. Demonstrate how a computer can measure the distance between two points using geometry, and then account for constraints (e.g., distance and then roads for directions) and multiple points with triangulation. Model what shapes and geometric relationships are most appropriate for different situations.
Exploring Changing Quantities	**Trig Explorations**	**G-SRT.1, G-SRT.2, G-SRT.3, G-SRT.5, G-SRT.9, G-SRT.10, G-SRT.11, GPE.7. G-C.2, G-C.4:** Investigate properties of right triangle similarity and congruence and the relationships between sine, cosine, and tangent; exploring the relationship between sine and cosine of complementary angles, and apply that knowledge to problem solving situations. Students recognize the role similarity plays in establishing trigonometric functions, and they use trigonometric functions to investigate situations. Using dynamic geometric software students investigate similarity and trigonometric identities to derive the Laws of Sines and Cosines and use the laws to solve problems.
Exploring Changing Quantities	**Triangle Problems**	**G-SRT.4, G-SRT.5, G-SRT.6, G-SRT.8, G-C.2, G-C.4, G-CO.12:** Understand and use congruence and similarity when solving problems involving triangles, including trigonometric ratios. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems using dynamic geometric software.

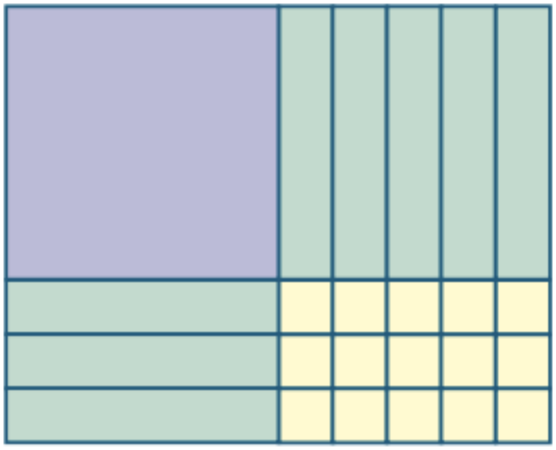
Content Connection	Big Idea	Geometry Standards
Exploring Changing Quantities	**Points & Shapes**	**G-GPE.1, G-GPE.2, G-GPE.4, G-GPE.5, G-GPE.6, G-GPE.7, G-CO.1, G-CO.12, G-C.2, G-C.4:** Solve problems involving geometric shapes in the coordinate plane using dynamic geometric software to apply the distance formula, Pythagorean Theorem, slope, and similarity rules in solving problems. • Investigate equations of circles and how coefficients in the equations correspond to the location and radius of the circles. Find areas and perimeters of triangles and rectangles in the coordinate plane.
Taking Wholes Apart, Putting Parts Together & Discovering Shape and Space	**Transformations**	**G-CO.1, G-CO.3, G-CO.4, G-CO.5, G-CO.12**: Understand rotations, reflections, and translations of regular polygons, quadrilaterals, angels, circles, and line segments. Identify transformations, through investigation, that move a figure back onto itself, using that process to prove congruence.
Discovering Shape and Space & Exploring Changing Quantities & Taking Wholes Apart, Putting Parts Together	**Triangle Congruence**	**G-CO.1, G-CO.2, G-CO.7, G-CO.8, G-CO.9, G-CO.10, G-CO.11, G-CO.12, G-CO.13, G-SRT.5:** Investigate triangles and their congruence over rigid transformations verifying findings using triangle congruence theorems (ASA, SSS, SAS, AAS, and HL) and other geometric properties, including vertical angles, angles created by transversals across parallel lines, and bisectors.

Content Connection	Big Idea	Geometry Standards
Exploring Changing Quantities & Discovering Shape and Space	**Circle Relationships**	**G-C.1, G-C.2, G-C.3, G-C.4, G-CO.1, G-CO.12, G-CO.13, G-GPE.1:** Investigate similarity in circles and relationships between angle measures and segments, including inscribed angles, radii, chords, central angles, inscribed angles, circumscribed angles, and tangent lines using dynamic geometric software.
Discovering Shape and Space	**Geometric Models**	**G-GMD.1, G-GMD.3, G-GMD.4, G-GMD.5, G-MG.1, G-MG.12, G-MG.13, SRT.5, G-CO.12, G-C.2, G-C.4:** Apply geometric concepts in modeling situations to solve design problems using dynamic geometric software. ● Investigate 3-D shapes and their cross sections. ● Use volume, area, circumference, and perimeter formulas. ● Understand and apply Cavalieri's principle. ● Investigate and apply scale factors for length, area, and volume.

Algebra I

The main purpose of Algebra I is to develop students' fluency with linear, quadratic, and exponential functions. The critical areas of instruction involve deepening and extending students' understanding of linear and exponential relationships by comparing and contrasting those relationships and by applying linear models to data that exhibit a linear trend. In addition, students engage in methods for analyzing, solving, and using exponential and quadratic functions. Some of the overarching elements of the Algebra I course include the notion of *function,* solving equations, rates of change and growth patterns, graphs as representations of functions, and modeling.

Figure A1-2. Algebra Tiles



The rectangle above has height $(x+3)$ and base $(x+5)$. The total area represented, the product of these binomials, is seen to be $\boldsymbol{x^2+5x+3x+15=x^2+8x+15}$.

For the Traditional Pathway, the standards in the Algebra I course come from the following conceptual categories: Modeling, Functions, Number and Quantity, Algebra, and Statistics and Probability. The course content is explained below according to these conceptual categories, but teachers and administrators alike should note that the standards are not listed here in the order in which they should be taught. Moreover, the standards are not simply topics to be checked off from a list during isolated units of instruction; rather, they represent content that should be present throughout the school year in rich instructional experiences.

Standards for Mathematical Practice *Students…*	Examples of each practice in Algebra I
MP1. Make sense of problems and persevere in solving them.	Students learn that patience is often required to fully understand what a problem is asking. They discern between what information is useful, and what is not. They expand their repertoire of expressions and functions that can used to solve problems.
MP2. Reason abstractly and quantitatively.	Students extend their understanding of slope as the rate of change of a linear function to understanding that the average rate of change of any function can be computed over an appropriate interval.
MP3. Construct viable arguments and critique the reasoning of others. ***Students build proofs by induction and proofs by contradiction. CA 3.1*** *(for higher mathematics only).*	Students reason through the solving of equations, recognizing that solving an equation is more than simply a matter of rote rules and steps. They use language such as "if… then..." when explaining their solution methods and provide justification.
MP4. Model with mathematics.	Students also discover mathematics through experimentation and examining patterns in data from real world contexts. Students apply their new mathematical understanding of exponential, linear and quadratic functions to real-world problems.
MP5. Use appropriate tools strategically.	Students develop a general understanding of the graph of an equation or function as a representation of that object, and they use tools such as graphing calculators or graphing software to create graphs in more complex examples, understanding how to interpret the result. They construct diagrams to solve problems.

Standards for Mathematical Practice *Students…*	Examples of each practice in Algebra I
MP6. Attend to precision.	Students begin to understand that a *rational number* has a specific definition, and that *irrational numbers* exist. They make use of the definition of *function* when deciding if an equation can describe a function by asking, "Does every input value have exactly one output value?"
MP7. Look for and make use of structure.	Students develop formulas such as $(a \pm b)^2 = a^2 \pm 2ab + b^2$ by applying the distributive property. Students see that the expression $5 + (x-2)^2$ takes the form of "5 plus 'something' squared", and so that expression can be no smaller than 5.
MP8. Look for and express regularity in repeating reasoning.	Students see that the key feature of a line in the plane is an equal difference in outputs over equal intervals of inputs, and that the result of evaluating the expression $\frac{y_2-y_1}{x_2-x_1}$ for points on the line is always equal to a certain number m. Therefore, if (x, y) is a generic point on this line, the equation $m = \frac{y-y_1}{x-x_1}$ will give a general equation of that line.

What Students Learn in Algebra I

In Algebra I, students use reasoning about structure to define and make sense of rational exponents and explore the algebraic structure of the rational and real number systems. They understand that numbers in real-world applications often have units attached to them—that is, the numbers are considered *quantities*.

Student work with numbers and operations throughout elementary and middle school leads them to an understanding of the structure of the number system; in Algebra I, students explore the structure of algebraic expressions and polynomials. They see that certain properties must persist when they work with expressions that are meant to represent numbers—which they now write in an abstract form involving variables. When two expressions with overlapping domains are set as equal to each other, resulting in an equation, there is an implied solution set (be it empty or non-empty), and students not only refine their techniques for solving equations and finding the solution set, but they can clearly explain the algebraic steps they used to do so.

Students began their exploration of linear equations in middle school, first by connecting proportional equations to graphs, tables, and real-world contexts, and then moving toward an understanding of general linear equations ($y = mx + b$, $m \neq 0$) and their graphs. In Algebra I, students extend this knowledge to work with absolute value equations, linear inequalities, and systems of linear equations. After learning a more precise definition of *function* in this course, students examine this new idea in the familiar context of linear equations—for example, by seeing the solution of a linear equation as solving for two linear functions.

Students continue to build their understanding of functions beyond linear types by investigating tables, graphs, and equations that build on previous understandings of numbers and expressions. They make connections between different representations of the same function. They also learn to build functions in a modeling context and solve problems related to the resulting functions. Note that in Algebra I the focus is on linear, simple exponential, and quadratic equations.

Finally, students extend their prior experiences with data, using more formal means of assessing how a model fits data. Students use regression techniques to describe approximately linear relationships between quantities. They use graphical representations and knowledge of the context to make judgments about the appropriateness of linear models. With linear models, students look at residuals to analyze the goodness of fit.

Examples of Key Advances from Kindergarten Through Grade Eight

- Having already extended arithmetic from whole numbers to fractions (grades four through six) and from fractions to rational numbers (grade seven), students in grade eight encountered specific irrational numbers such as 5 and ð. In Algebra I, students begin to understand the real number *system*. See Chapter Three: Number Sense for a detailed progression of how students' understanding of numbers develops through the grades.
- Students in middle grades worked with measurement units, including units obtained by multiplying and dividing quantities. In Algebra I (conceptual category N–Q), students apply these skills in a more sophisticated fashion to solve problems in which reasoning about units adds insight.

- Algebraic themes beginning in middle school continue and deepen during high school. As early as grades six and seven, students began to use the properties of operations to generate equivalent expressions (standards 6.EE.3 and 7.EE.1). By grade seven, they began to recognize that rewriting expressions in different forms could be useful in problem solving (standard 7.EE.2). In Algebra I, these aspects of algebra carry forward as students continue to use properties of operations to rewrite expressions, gaining fluency and engaging in what has been called "mindful manipulation."
- Students in grade eight extended their prior understanding of proportional relationships to begin working with functions, with an emphasis on linear functions. In Algebra I, students learn linear and quadratic functions. Students encounter other kinds of functions to ensure that general principles of working with functions are perceived as applying to all functions, as well as to enrich the range of quantitative relationships considered in problems.
- Students in grade eight connected their knowledge about proportional relationships, lines, and linear equations (standards 8.EE.5–6). In Algebra I, students solidify their understanding of the analytic geometry of lines. They understand that in the Cartesian coordinate plane: the graph of any linear equation in two variables is a line; any line is the graph of a linear equation in two variables.
- As students acquire mathematical tools from their study of algebra and functions, they apply these tools in statistical contexts (e.g., standard S-ID.6). In a modeling context, they might informally fit a quadratic function to a set of data, graphing the data and the model function on the same coordinate axes. They also draw on skills first learned in middle school to apply basic statistics and simple probability in a modeling context. For example, they might estimate a measure of center or variation and use it as an input for a rough calculation.
- Algebra I techniques open an extensive variety of solvable word problems that were previously inaccessible or very complex for students in kindergarten through grade eight. This expands problem solving dramatically.

Example:

Information	Teacher Moves
Exponential Growth. When a quantity grows with time by a multiplicative factor greater than 1, it is said the quantity grows exponentially. Hence, if an initial population of bacteria, P_0, doubles each day, then after t days, the new population is given by $P(t) = P_0 2^t$. This expression can be generalized to include different growth rates, r, as in $P(t) = P_0 r^t$. The following example illustrates the type of problem that students can face after they have worked with basic exponential functions like these. **Example.** On June 1, a fast-growing species of algae is accidentally introduced into a lake in a city park. It starts to grow and cover the surface of the lake in such a way that the area covered by the algae doubles every day. If it continues to grow unabated, the lake will be totally covered and the fish in the lake will suffocate. At the rate it is growing, this will happen on June 30.	**Possible Questions to Ask:** a. When will the lake be covered halfway? b. Write an equation that represents the percentage of the surface area of the lake that is covered in algae as a function of time (in days) that passes since the algae was introduced into the lake. **Solution and Comment.** a. Since the population doubles each day, and since the entire lake is covered by June 30, this implies that half the lake was covered on June 29. b. If $P(t)$ represents the *percentage* of the lake covered by the algae, then we know that $P(29) = P_0 2^{29} = 100$ (note that June 30 corresponds to $t = 29$). Therefore, we can solve for the initial percentage of the lake covered, $P_0 = \frac{100}{2^{29}} \approx 1.86 \times 10^{-7}$. The equation for the percentage of the lake covered by algae at time t is therefore $P(t) = (1.86 \times 10^{-7})2^t$.

Geometry

The fundamental purpose of the geometry course is to introduce students to formal geometric proofs and the study of plane figures, culminating in the study of right-triangle trigonometry and circles. Students begin to formally prove results about the geometry of the plane by using previously defined terms and notions. Similarity is explored in greater detail, with an emphasis on discovering trigonometric relationships and solving problems with right triangles. The correspondence between the plane and the Cartesian coordinate system is explored when students connect algebra concepts with geometry concepts. Students explore probability concepts and use probability in real-world situations. The major mathematical ideas in the geometry course include geometric transformations, proving geometric theorems, congruence and similarity, analytic geometry, right-triangle trigonometry, and probability.

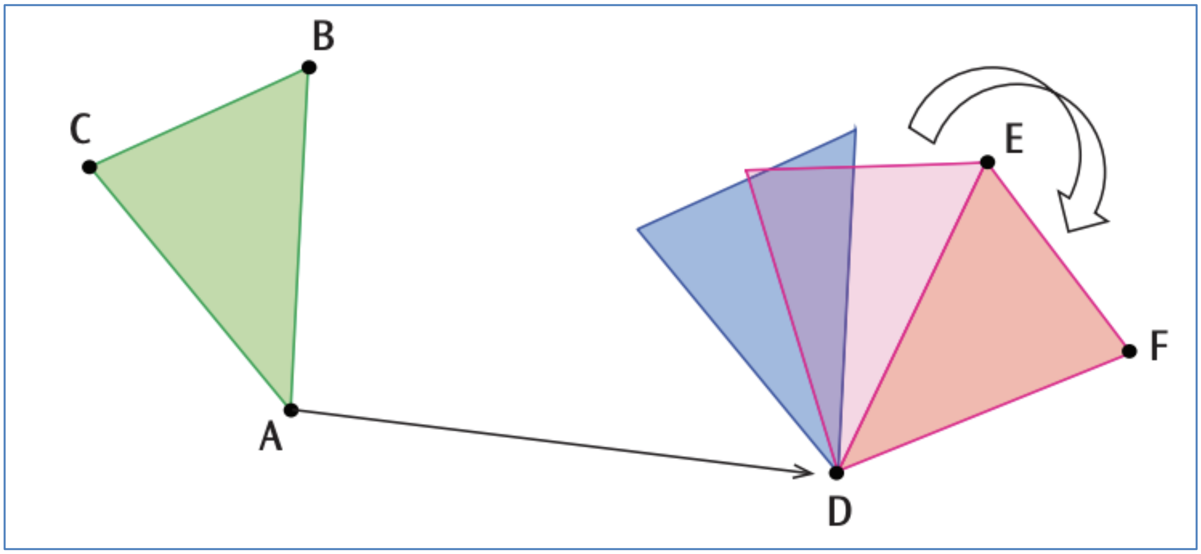
Standards for Mathematical Practice *Students…*	Examples of each practice in Geometry
MP1. Make sense of problems and persevere in solving them.	Students construct accurate diagrams of geometry problems to help make sense of them. They organize their work so that others can follow their reasoning, e.g., in proofs.
MP2. Reason abstractly and quantitatively.	Students understand that the coordinate plane can be used to represent geometric shapes and transformations and therefore connect their understanding of number and algebra to geometry.
MP3. Construct viable arguments and critique the reasoning of others. ***Students build proofs by induction and proofs by contradiction. CA 3.1*** *(for higher mathematics only).*	Students construct proofs of geometric theorems. They write coherent logical arguments and understand that each step in a proof must follow from the last, justified with a previously accepted or proven result.
MP4. Model with mathematics.	Students apply their new mathematical understanding to real-world problems. They learn how transformational geometry and trigonometry can be used to model the physical world.
MP5. Use appropriate tools strategically.	Students make use of visual tools for representing geometry, such as simple patty paper or transparencies, or dynamic geometry software.

Standards for Mathematical Practice *Students…*	Examples of each practice in Geometry
MP6. Attend to precision.	Students develop and use precise definitions of geometric terms. They verify that a specific shape has certain properties justifying its categorization (e.g., a rhombus as opposed to a quadrilateral).
MP7. Look for and make use of structure.	Students construct triangles in quadrilaterals or other shapes and use congruence criteria of triangles to justify results about those shapes.
MP8. Look for and express regularity in repeated reasoning.	Students explore rotations, reflections and translations, noticing that certain attributes of different shapes remain the same (e.g., parallelism, congruency, orientation) and develop properties of transformations by generalizing these observations.

The standards in the traditional geometry course come from the following conceptual categories: Modeling, Geometry, and Statistics and Probability. The content of the course is explained below according to these conceptual categories, but teachers and administrators alike should note that the standards are not listed here in the order in which they should be taught. Moreover, the standards are not topics to be checked off after being covered in isolated units of instruction; rather, they provide content to be developed throughout the school year through rich instructional experiences.

What Students Learn in Geometry

Although there are many types of geometry, school mathematics is devoted primarily to plane Euclidean geometry, studied both synthetically (without coordinates) and analytically (with coordinates). In the higher mathematics courses, students begin to formalize their geometry experiences from elementary and middle school, using definitions that are more precise and developing careful proofs. The standards for grades seven and eight call for students to see two-dimensional shapes as part of a generic plane (i.e., the Euclidean plane) and to explore transformations of this plane as a way to determine whether two shapes are congruent or similar.



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These concepts are formalized in the geometry course, and students use transformations to prove geometric theorems. The definition of congruence in terms of rigid motions provides a broad understanding of this means of proof, and students explore the consequences of this definition in terms of congruence criteria and proofs of geometric theorems.

Students investigate triangles and decide when they are similar—and with this newfound knowledge and their prior understanding of proportional relationships, they define trigonometric ratios and solve problems by using right triangles. They investigate circles and prove theorems about them. Connecting to their prior experience with the coordinate plane, they prove geometric theorems by using coordinates and describe shapes with equations. Students extend their knowledge of area and volume formulas to those for circles, cylinders, and other rounded shapes. Finally, continuing the development of statistics and probability, students investigate probability concepts in precise terms, including the independence of events and conditional probability.

Examples of Key Advances from Previous Grade Levels or Courses

- Because concepts such as rotation, reflection, and translation were treated in the grade-eight standards mostly in the context of hands-on activities and with an emphasis on geometric intuition, the geometry course places equal weight on precise definitions.

- In kindergarten through grade eight, students worked with a variety of geometric measures: length, area, volume, angle, surface area, and circumference. In geometry, students apply these component skills in tandem with others in the course of modeling tasks and other substantial applications (MP.4).
- The skills that students develop in Algebra I around simplifying and transforming square roots will be useful when solving problems that involve distance or area and that make use of the Pythagorean Theorem.
- Students in grade eight learned the Pythagorean Theorem and used it to determine distances in a coordinate system (8.G.6–8). In geometry, students build on their understanding of distance in coordinate systems and draw on their growing command of algebra to connect equations and graphs of circles (G-GPE.1).
- The algebraic techniques developed in Algebra I can be applied to study analytic geometry. Geometric objects can be analyzed by the algebraic equations that give rise to them. Algebra can be used to prove some basic geometric theorems in the Cartesian plane.

Example: Defining Rotations

Mrs. B wants to help her class understand the following definition of a rotation:
A rotation about a point P through angle α is a transformation A⟼A^' such that (1) if point A is different from P, then PA=PA^' and the measure of ∠APA^'=α; and (2) if point A is the same as point P, then A^'=A.

She gives her students a handout with several geometric shapes on it and a point P indicated on the page. In pairs, students are to copy the shapes onto a transparency sheet and rotate them through various angles about P. Students then transfer the rotated shapes back onto the original page, and measure various lengths and angles as indicated in the definition. While justifying that the properties of the definition hold for the shapes she has given them, the students also make some observations about the effects of a rotation on the entire plane, for instance that:

Rotations preserve lengths.

Rotations preserve angle measures.

Rotations preserve parallelism.

Later, Mrs. B plans to allow students to explore more rotations on dynamic geometry software, asking them to create a geometric shape and rotate it by various angles about various points P, both part of the object and not.

Algebra II

Algebra II course extends students' understanding of functions and real numbers and increases the tools students have for modeling the real world. Students in Algebra II extend their notion of number to include complex numbers and see how the introduction of this set of numbers yields the solutions of polynomial equations and the Fundamental Theorem of Algebra. Students deepen their understanding of the concept of function and apply equation-solving and function concepts to many different types of functions. The system of polynomial functions, analogous to integers, is extended to the field of rational functions, which is analogous to rational numbers. Students explore the relationship between exponential functions and their inverses, the logarithmic functions. Trigonometric functions are extended to all real numbers, and their graphs and properties are studied. Finally, students' knowledge of statistics is extended to include under- standing the normal distribution, and students are challenged to make inferences based on sampling, experiments, and observational studies.

For the Traditional Pathway, the standards in the Algebra II course come from the following conceptual categories: Modeling, Functions, Number and Quantity, Algebra, and Statistics and Probability. The course content is explained below according to these conceptual categories, but teachers and administrators alike should note that the standards are not listed here in the order in which they should be taught. Moreover, the standards are not simply topics to be checked off from a list during isolated units of instruction; rather, they represent content that should be present throughout the school year in meaningful and rigorous instructional experiences.

Standards for Mathematical Practice *Students…*	Examples of each practice in Algebra II
MP1. Make sense of problems and persevere in solving them.	Students apply their understanding of various functions to real-world problems. They approach complex mathematics problems and break them down into smaller-sized chunks and synthesize the results when presenting solutions.
MP2. Reason abstractly and quantitatively.	Students deepen their understanding of variable, for example, by understanding that changing the values of the parameters in the expression $A\sin(Bx+C)+D$ has consequences for the graph of the function. They interpret these parameters in a real-world context.
MP3. Construct viable arguments and critique the reasoning of others. ***Students build proofs by induction and proofs by contradiction. CA 3.1*** *(for higher mathematics only).*	Students continue to reason through the solution of an equation and justify their reasoning to their peers. Students defend their choice of a function to model a real-world situation.
MP4. Model with mathematics.	Students apply their new mathematical understanding to real-world problems, making use of their expanding repertoire of functions in modeling. Students also discover mathematics through experimentation and examining patterns in data from real world contexts.
MP5. Use appropriate tools strategically.	Students continue to use graphing technology to deepen their understanding of the behavior of polynomial, rational, square root, and trigonometric functions.
MP6. Attend to precision.	Students make note of the precise definition of *complex number,* understanding that real numbers are a subset of the complex numbers. They pay attention to units in real-world problems and use unit analysis as a method for verifying their answers.

Standards for Mathematical Practice *Students…*	Examples of each practice in Algebra II
MP7. Look for and make use of structure.	Students see the operations of the complex numbers as extensions of the operations for real numbers. They understand the periodicity of sine and cosine and use these functions to model periodic phenomena.
MP8. Look for and express regularity in repeating reasoning.	Students observe patterns in geometric sums, e.g., that the first several sums of the form $\sum_{k=0}^{n} 2^k$ can be written: $1 = 2^1 - 1;$ $1 + 2 = 2^2 - 1;$ $1 + 2 + 4 = 2^3 - 1;$ $1 + 2 + 4 + 8 = 2^4 - 1;$ and use this observation to make a conjecture about any such sum.

What Students Learn in Algebra II

Building on their work with linear, quadratic, and exponential functions, students in Algebra II extend their repertoire of functions to include polynomial, rational, and radical functions.

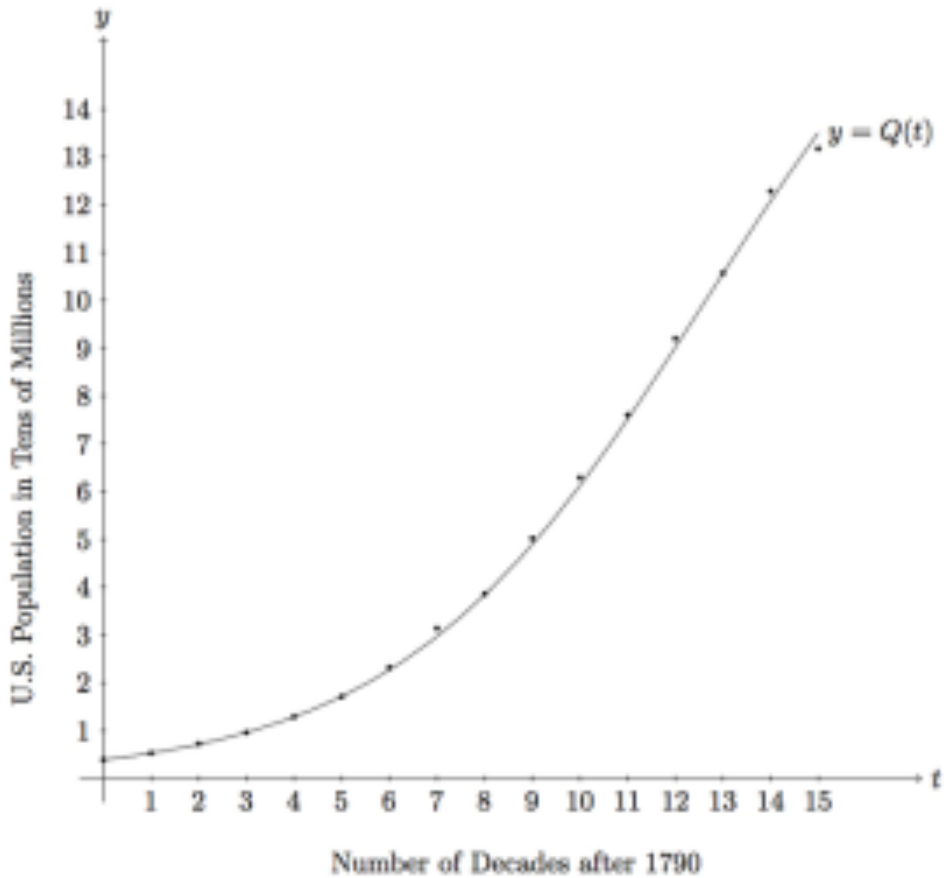
Students work closely with the expressions that define the functions and continue to expand and hone their abilities to model situations and to solve equations, including solving quadratic equations over the set of complex numbers and solving exponential equations using the properties of logarithms. Based on their previous work with functions, and on their work with trigonometric ratios and circles in geometry, students now use the coordinate plane to extend trigonometry to model periodic phenomena. They explore the effects of transformations on graphs of diverse functions, including functions arising in applications, in order to abstract the general principle that transformations on a graph always have the same effect regardless of the type of underlying function. They identify appropriate types of functions to model a situation, adjust parameters to improve the model, and compare models by analyzing appropriateness of fit and making judgments about the domain over which a model is a good fit.

Example (Adapted from Illustrative Mathematics, 2013)

Population Growth. The approximate United States Population measured each decade starting in 1790 up through 1940 can be modeled by the function

$$P(t) = \frac{(3{,}900{,}000 \times 200{,}000{,}000)e^{0.31t}}{200{,}000{,}000 + 3{,}900{,}000(e^{0.31t} - 1)}$$

where t represents decades after 1790. Such models are important for planning infrastructure and the expansion of urban areas, and historically accurate long-term models have been difficult to derive.



Some possible questions:

a. According to this model for the U.S. population, what was the population in the year 1790?

b. According to this model, when did the population first reach 100,000,000? Explain.

c. According to this model, what should be the population of the U.S. in the year 2010? Find a prediction of the U.S. population in 2010 and compare with your result.

d. For larger values of t, such as $t = 50$, what does this model predict for the U.S. population? Explain your findings.

Solutions:

a) The population in 1790 is given by $P(0)$, which we easily find is $3{,}900{,}000$ since $e^{0.31(0)} = 1$.

b) This is asking us to find t such that $P(t) = 100{,}000{,}000$. Dividing the numerator and denominator on the left by 1,000,000 and dividing both sides of the equation by $100{,}000{,}000$ simplifies this equation to

$$\frac{3.9 \times 2 \times e^{.31t}}{200 + 3.9(e^{.31t} - 1)} = 1.$$

Using some algebraic manipulation and solving for t gives $t \approx \frac{1}{0.31} \ln 50.28 \approx 12.64$.

This means it would take about 126.4 years after 1790 for the population to reach 100 million.

c) The population 22 decades after 1790 would be approximately 190,000,000, too low by about 119,000,000 from the estimated U.S. population of 309,000,000 in 2010.

d) The structure of the expression reveals that for very large values of t, the denominator is dominated by $3{,}900{,}000e^{.31t}$. Thus, for very large t,

$$P(t) \approx \frac{3{,}900{,}000 \times 200{,}000{,}000 \times e^{.31t}}{3{,}900{,}000e^{.31t}} = 200{,}000{,}000$$

Therefore, the model predicts a population that stabilizes at 200,000,000 as t increases.

Students see how the visual displays and summary statistics learned in earlier grade levels relate to different types of data and to probability distributions. They identify different ways of collecting data—including sample surveys, experiments, and simulations—and the role of randomness and careful design in the conclusions that can be drawn.

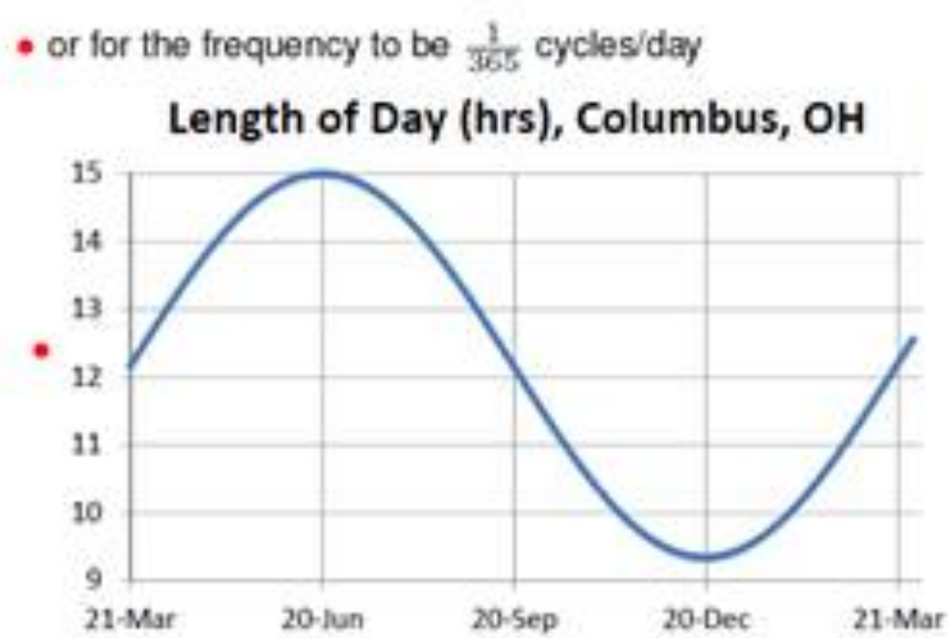
Examples of Key Advances from Previous Grade Levels or Courses

- In Algebra I, students added, subtracted, and multiplied polynomials. Students in Algebra II divide polynomials that result in remainders, leading to the factor and remainder theorems. This is the underpinning for much of advanced algebra, including the algebra of rational expressions.
- Themes from middle-school algebra continue and deepen during high school. As early as grade six, students began thinking about solving equations as a process of reasoning (6.EE.5). This perspective continues throughout Algebra I and Algebra II (A-REI). "Reasoned solving" plays a role in Algebra II because the equations students encounter may have extraneous solutions (A-REI.2).
- In Algebra I, students worked with quadratic equations with no real roots. In Algebra II, they extend their knowledge of the number system to include complex numbers, and one effect is that they now have a complete theory of quadratic equations: Every quadratic equation with complex coefficients has (counting multiplicity) two roots in the complex numbers.
- In grade eight, students learned the Pythagorean Theorem and used it to determine distances in a coordinate system (8.G.6–8). In the geometry course, students proved theorems using coordinates (G-GPE.4–7). In Algebra II, students build on their understanding of distance in coordinate systems and draw on their growing command of algebra to connect equations and graphs of conic sections (for example, refer to standard G-GPE.1).
- In geometry, students began trigonometry through a study of right triangles. In Algebra II, they extend the three basic functions to the entire unit circle.
- As students acquire mathematical tools from their study of algebra and functions, they apply these tools in statistical contexts (for example, refer to standard S-ID.6). In a modeling context, students might informally fit an exponential function to a set of data, graphing the data and the model function on the same coordinate axes (Partnership for Assessment of Readiness for College and Careers 2012).

Example (from Achieve the Core, 2013, 19)

Modeling Daylight Hours. By looking at data for length of days in Columbus, OH, students see that day length is approximately sinusoidal, varying from about 9 hours, 20 minutes on December 21 to about 15 hours on June 21. The average of the maximum and minimum gives the value for the midline, and the amplitude is half the different of the maximum and minimum. They set $A = 12.17$ and $B = 2.83$ as approximations of these values. With some support, students determine that for the period to be 365 days (per cycle), $C = 2\pi/365$ and if day 0 corresponds to March 21, no phase shift would be needed, so $D = 0$.

Thus, $f(t) = 12.17 + 2.83\sin\left(\frac{2\pi t}{365}\right)$ is a function that gives the approximate length of day for t the day of the year from March 21. Considering questions such as when to plant a garden, i.e., when there are at least 7 hours of midday sunlight, students might estimate that a 14-hour day is optimal. Students solve $f(t) = 14$, and find that May 1 and August 10 bookend this interval of time.



Students can investigate many other trigonometric modeling situations such as simple predator-prey models, sound waves, and noise cancellation models.

The Integrated Mathematics Pathway

Many schools and districts in California have implemented an “Integrated Mathematics Pathway” according to the course outlines in the CA CCSSM. In recognition of this investment, this Framework continues to support these pathways, as the field strives to develop truly integrated approaches (in the sense of the *Definition of Integration* in Chapter 8) to the teaching and learning of higher mathematics content. The standards for the Integrated Pathway, by course, begin on page 85 of the CA CCSSM (CDE, 2013).

These courses are described here.

Integrated Pathway Big Ideas

The state of California set out the most important mathematical content and practices by highlighting a collection of big ideas in mathematics, TK–10 in the Digital Learning and Standards Initiative (CDE, 2021). In this document, the CA CCSSM content standards and Standards for Mathematical Practice in transitional kindergarten through grade ten were organized into a set of Big Ideas, which themselves are organized into the Content Connections.

Figure A.6 presents the progression of Big Ideas for the Integrated 1 and 2 course sequence. The network maps, in Figures A.7 and A.9, highlight important and foundational content, shown as nodes, for each grade level. As students explore and investigate with the Big Ideas, they will likely encounter many different content standards and note the connections between them. The size of a node relates to the number of connections it has with other Big Ideas. The connections between Big Ideas are made when the two connected Big Ideas contain one or more of the same standards.

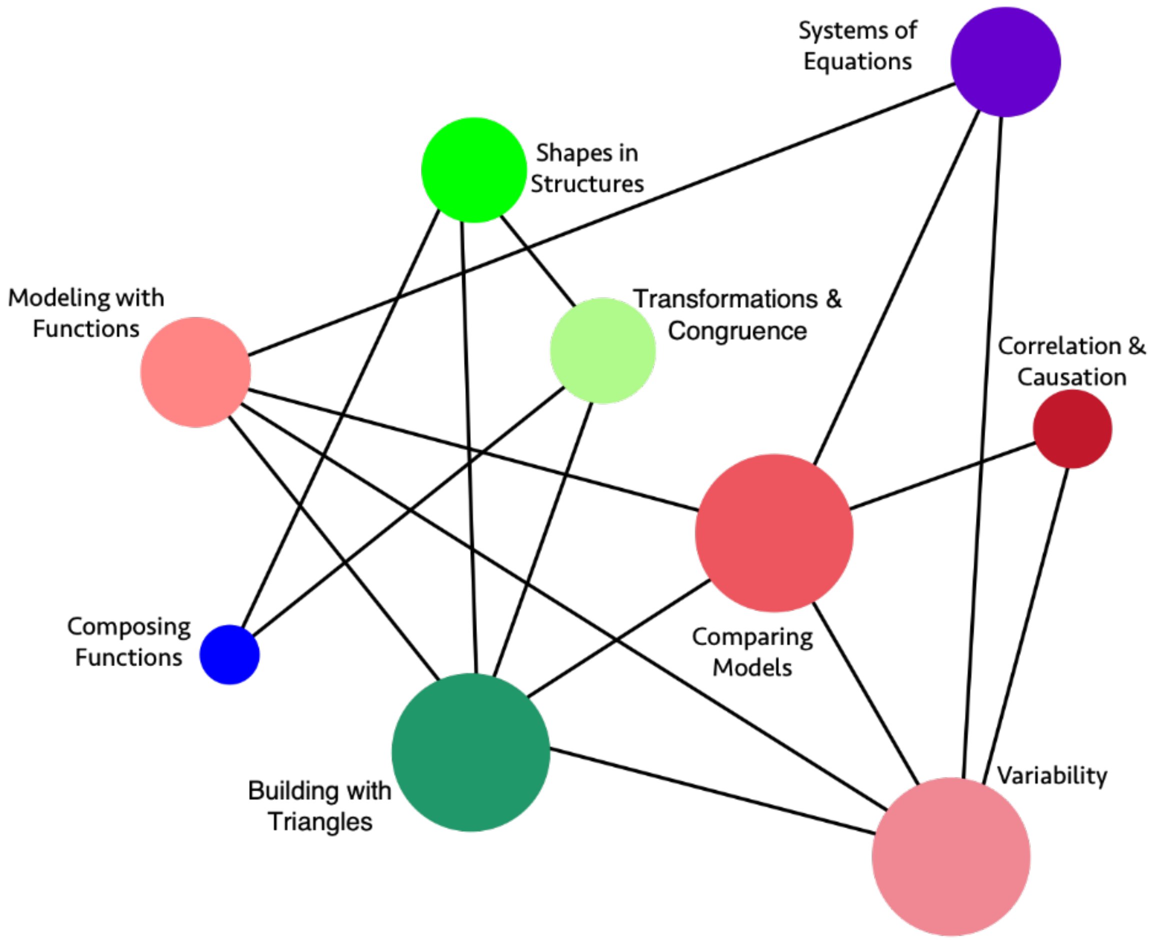
The colors in the nodes correspond to the Content Connections, Big Ideas, and Standards tables, Figures A.8 and A.10, which follow each of the network diagrams for the two courses. The Big Ideas (middle column) are situated within their broader Content Connection (left column), and the CA CCSSM content standards (right column) which can be addressed for each Big Idea are indicated.

Figure A.6: A Progression Chart of Big Ideas through Integrated 1 and 2

Content Connections	**Big Ideas: Integrated 1**	**Big Ideas: Integrated 2**
Communicating Stories with Data	Modeling with functions	The shape of distributions
Communicating Stories with Data	Comparing models	Geospatial data
Communicating Stories with Data	Variability	Probability modeling
Communicating Stories with Data	Correlation & causation	Experimental models and functions
Exploring Changing Quantities	Modeling with functions	The shape of distributions
Exploring Changing Quantities	Comparing models	Equations to predict & model
Exploring Changing Quantities	Variability	Experimental models & functions
Exploring Changing Quantities	Systems of equations	Transformation & similarity
Taking Wholes Apart, Putting Parts Together	Systems of equations	Functions in the world
Taking Wholes Apart, Putting Parts Together	Composing functions	Polynomial identities
Taking Wholes Apart, Putting Parts Together	Shapes in structures	Function representations
Taking Wholes Apart, Putting Parts Together	Building with triangles	n/a
Discovering shape and space	Shapes in structures	Circle relationships
Discovering shape and space	Building with triangles	Trig functions

Content Connections	Big Ideas: Integrated 1	Big Ideas: Integrated 2
Discovering shape and space	Transformations & congruence	Transformation & similarity

Figure A.7: High School Integrated 1 Big Ideas



[Link to long description]

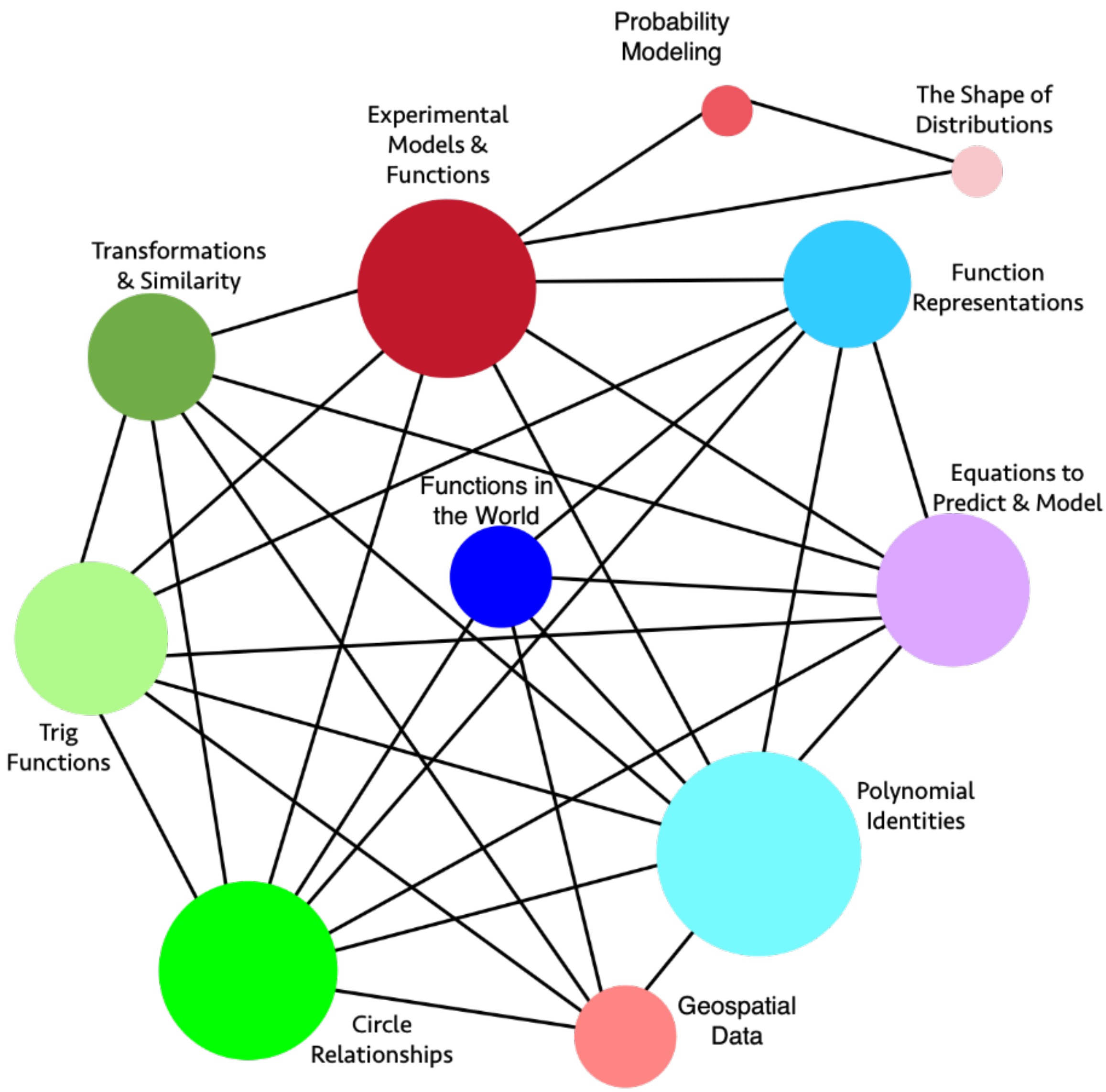
Figure A.8: High School Integrated 1 Content Connections, Big Ideas, and Standards

Content Connection	Big Idea	Integrated 1 Standards
Communicating Stories with Data & Exploring Changing Quantities	**Modeling with Functions**	**N-Q.1, N-Q.2, N-Q.3, A-CED.2, F-BF.1, F-IF.1, F-IF.2, F-IF.4, F-LE.5, S-ID.7, A-CED.1, A-CED.2, A-CED.3, A-SSE.1:** Build functions that model relationships between two quantities, including examples with inequalities; using units and different representations. Describe and interpret the relationships modeled using visuals, tables, and graphs.
Communicating Stories with Data & Exploring Changing Quantities	**Comparing Models**	**F-LE.1, F-LE.2, F-LE.3, F-IF.4, F-BF.1, F-LE.5, S-ID.7, S-ID.8, A-CED.1, A-CED.2, A-CED.3, A-SSE.1:** Construct, interpret, and compare linear, quadratic, and exponential models of real data, and use them to describe and interpret the relationships between two variables, including inequalities. Interpret the slope and constant terms of linear models, and use technology to compute and interpret the correlation coefficient of a linear fit.
Communicating Stories with Data & Exploring Changing Quantities	**Variability**	**S-ID.5, S-ID.6, S-ID.7, S-ID.1, S-ID.2, S-ID.3, S-ID.4, A-SSE.1:** Summarize, represent, and interpret data. For quantitative data, use a scatter plot and describe how the variables are related. Summarize categorical data in two-way frequency tables and interpret the relative frequencies.
Communicating Stories with Data	**Correlation & Causation**	**S-ID.9, S-ID.8, S-ID.7:** Explore data that highlights the difference between correlation and causation. Understand and use correlation coefficients, where appropriate. (see resource section for classroom examples).

Content Connection	Big Idea	Integrated 1 Standards
Exploring Changing Quantities & Taking Wholes Apart, Putting Parts Together	**Systems of Equations**	**A-REI.1, A-REI.3, A-REI.4, A-REI.5, A-REI.6, A-REI.7, A-REI.10, A-REI.11, A-REI.12, NQ.1, A-SEE.1:** Students investigate real situations that include data for which systems of 1 or 2 equations or inequalities are helpful, paying attention to units. Investigations include linear, quadratic, and absolute value. Students use technology tools strategically to find their solutions and approximate solutions, constructing viable arguments, interpreting the meaning of the results, and communicating them in multidimensional ways.
Taking Wholes Apart, Putting Parts Together	**Composing Functions**	**F-BF.3, F-BF.2, F-IF.3:** Build and explore new functions that are made from existing functions, and explore graphs of the related functions using technology. Recognize sequences are functions and are defined recursively.
Taking Wholes Apart, Putting Parts Together & Discovering Shape and Space	**Shapes in Structures**	**G-CO.6, C-CO.7, C-CO.8, G-GPE.4, G-GPE.5, G.GPE.7, F.BF.3:** Perform investigations that involve building triangles and quadrilaterals, considering how the rigidity of triangles and non-rigidity of quadrilaterals influences the design of structures and devices. Study the changes in coordinates and express the changes algebraically.
Taking Wholes Apart, Putting Parts Together & Discovering Shape and Space	**Building with Triangles**	**G-GPE.4, G-GPE.5, G-GPE.6, GPE.7, F-LE.1, F-LE.2, A-CED.2:** Investigate with geometric figures, constructing figures in the plane, relating the distance formula to the Pythagorean Theorem, noticing how areas and perimeters of polygons change as the coordinates change. Build with triangles and quadrilaterals, noticing positions and movement, and creating equations that model the changing edges using technology.

Content Connection	Big Idea	Integrated 1 Standards
Discovering Shape and Space	**Transformations & Congruence**	**G-CO.1, G-CO.2, G-CO.3, G-CO.4, G-CO.5, G-CO.12, G-CO.13, G-GPE.4, G-GPE.5, G.GPE.7, F-BF.3:** Explore congruence of triangles, including quadrilaterals built from triangles, through geometric constructions. Investigate transformations in the plane. Use geometry software to study transformations, developing definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, and parallel lines. Express translations algebraically.

Figure A.9: Big Ideas Map for Integrated 2



[Link to long description]

Figure A.10: High School Integrated 2 Content Connections, Big Ideas, and Standards

Content Connection	Big Idea	Integrated 2 Standards
Communicating Stories with Data	**Probability Modeling**	**S.CP.1, S.CP.2, S.CP.3, S.CP.4, S.CP.5, S-IC.1, S-IC.2, S-IC.3, S.MD.6, S.MD.7:** Explore and compare independent and conditional probabilities, interpreting the output in terms of the model. Construct and interpret two-way frequency tables of data as a sample space to determine if the events are independent, and use the data to approximate conditional probabilities. Examples of topics include product and medical testing, and player statistics in sports.
Communicating Stories with Data	**The shape of distributions**	**S-IC.1, S-IC.2, S-IC.3, S-ID.1, S-ID.2, S-ID.3, S-MD.1, S-MD.2:** Consider the shape of data distributions to decide on ways to compare the center and spread of data. Use simulation models to generate data, and decide if the model produces consistent results.
Communicating Stories with Data & Exploring Changing Quantities	**Experimental Models & Functions**	**S-ID.1, S-ID.2, S-ID.3, S- ID.6, S-ID.7, S-IC.1, S-IC.2, S-IC.3, A-CED.1, A-REI.1, A-REI.4, F-IF.2, F-IF.3, F-IF.4, F-BF.1, F-LE.1, F-TF.2, A-APR.1:** Conduct surveys, experiments, and observational studies - drawing conclusions and making inferences. Compare different data sources and what may be most appropriate for the situation. Create and interpret functions that describe the relationships, interpreting slope and the constant term when linear models are used. Include quadratic and exponential models when appropriate, and understand the meaning of outliers.
Communicating Stories with Data	**Geospatial Data**	**G-MG.1, G-MG.2, G-MG.3, F-LE.6, G-GPE.4, G-GPE.6, G-SRT.5, G-CO.1, G-CO.2, G-CO.12, G-C.2, G-C.5:** Explore geospatial data that represent either locations (e.g., maps) or objects (e.g., patterns of people's faces, road objects for driverless cars) and connect to geometric equations and properties of common shapes. Demonstrate how a computer can measure the distance between two points using geometry and then account for constraints (e.g., distance and then roads for directions) and multiple points with triangulation. Model what shapes and geometric relationships are most appropriate for different situations.

Content Connection	Big Idea	Integrated 2 Standards
Exploring Changing Quantities	**Equations to Predict & Model**	**A-CED.1, A-CED.2, A-REI.4, A-REI.1, A-REI.2, A-REI.3, F.IF.4, F.IF.5, F.IF.6, F.BF.1, F.BF.3, A-APR.1:** Model relationships that include creating equations or inequalities, including linear, quadratic, and absolute value. Use the equations or inequalities to make sense of the world or to make predictions, understanding that solving equations is a process of reasoning. Make sense of the real situation, using multiple representations, such as graphs, tables, and equations.
Taking Wholes Apart, Putting Parts Together	**Functions in the World**	**F-LE.3, F-LE.6, F-IF.9, N-RN.1**, **N-RN.2**, **A-SSE.1**, **A-SSE.2:** Apply quadratic functions to the physical world, such as motion of an object under the force of gravity. Produce equivalent forms of the functions to reveal zeros, max and min, and intercepts. Investigate how functions increase and decrease, and compare the rates of increase or decrease to linear and exponential functions.
Taking Wholes Apart, Putting Parts Together	**Polynomial Identities**	**A-SSE.1, A-SSE.2, A-APR.1, A-APR.3, A-APR.4, G-GMD.2, G-MG.1, S-IC.1, S-MD.2:** Prove polynomial identities, and use them to describe numerical relationships, using a computer algebra system to rewrite polynomials. Use the binomial theorem to solve problems, appreciating the connections with Pascal's triangle.
Taking Wholes Apart, Putting Parts Together	**Functions Representations**	**F-IF.4, F-IF.5, F-IF.6, F-IF.7, F-IF.8, F-IF.9, N-RN.1, N-RN.2, F-LE.3, A-APR.1:** Interpret functions representing real world applications in terms of the data understanding key features of graphs, tables, domain, and range. Compare properties of two functions each represented in different ways (algebraically, graphically, numerically, in tables or by written/verbal descriptions).

Content Connection	Big Idea	Integrated 2 Standards
Discovering Shape and Space & Exploring Changing Quantities	**Transformations & Similarity**	**G-SRT.1, G- SRT.2, G-SRT.3, , A-CED.2, G-GPE.4, F-BF.3, F-IF.4, A-APR.1:** Explore similarity and congruence in terms of transformations, noticing the changes dilations have on figures and the effect of scale factors. Discover how coordinates can be used to describe translations, rotations, and reflections, and generalize findings to model the transformations using algebra.
Discovering Shape and Space	**Circle Relationships**	**G-C.1, G-C.2, G-C.3, G-C.4, G-C.5, G-GPE.1, A-REI.7, A-APR.1, F-IF.9:** Investigate the relationships of angles, radii, and chords in circles, including triangles and quadrilaterals that are inscribed and circumscribed. Explore arc lengths and areas of sectors using the coordinate plane. Relate the Pythagorean Theorem to the equation of the circle given the center and radius, and solve simple systems where a line intersects the circle.
Discovering Shape and Space	**Trig Functions**	**G-TF.2, G-GPE.1, G-GMD.2, G-MG.1, A-APR.1:** Model periodic phenomena with trigonometric functions. Translate between geometric descriptions and the equation for a conic section. Visualize relationships between 2-D and 3-D objects.

Integrated Math I

The fundamental purpose of the Mathematics I course is to formalize and extend students' understanding of linear functions and their applications. The critical topics of study deepen and extend understanding of linear relationships—in part, by contrasting them with exponential phenomena and, in part, by applying linear models to data that exhibit a linear trend. Mathematics I uses properties and theorems involving congruent figures to deepen and extend geometric knowledge gained in prior grade levels. The courses in the Integrated Pathway follow the structure introduced in the K–8 grade levels of the CA CCSSM; they present mathematics as a coherent subject and blend standards from different conceptual categories.

The standards in the integrated Mathematics I course come from the following conceptual categories: Modeling, Functions, Number and Quantity, Algebra, Geometry, and Statistics and Probability. The content of the course is explained in the addendum according to these conceptual categories, but teachers and administrators alike should note that the standards are not listed here in the order in which they should be taught. Moreover, the standards are not topics to be checked off after being covered in isolated units of instruction; rather, they provide content to be developed throughout the school year through rich instructional experiences.

What Students Learn in Mathematics I

Students in Mathematics I continue their work with expressions and modeling and analysis of situations. In previous grade levels, students informally defined, evaluated, and compared functions, using them to model relationships between quantities. In Mathematics I, students learn function notation and develop the concepts of domain and range. Students move beyond viewing functions as processes that take inputs and yield outputs and begin to view functions as objects that can be combined with operations (e.g., finding). They explore many examples of functions, including sequences. They interpret functions that are represented graphically, numerically, symbolically, and verbally, translating between representations and understanding the limitations of various representations. They work with functions given by graphs and tables, keeping in mind that these representations are likely to be approximate and incomplete, depending upon the context. Students' work includes functions that can be described or approximated by formulas, as well as those that cannot. When functions describe relationships between quantities arising from a context, students reason with the units in which those quantities are measured. Students build on and informally extend their understanding of integer exponents to consider exponential functions. They compare and contrast linear and exponential functions, distinguishing between additive and multiplicative change. They also interpret arithmetic sequences as linear functions and geometric sequences as exponential functions.

Students who are prepared for Mathematics I have learned to solve linear equations in one variable and have applied graphical and algebraic methods to analyze and solve systems of linear equations in two variables. Mathematics I builds on these earlier experiences by asking students to analyze and explain the process of solving an equation and to justify the process used in solving a system of equations. Students develop fluency in writing, interpreting, and translating between various forms of linear equations and inequalities and using them to solve problems. They master solving linear equations and apply related solution techniques and the laws of exponents to the creation and solving of simple exponential equations. Students explore systems of equations and inequalities, finding and interpreting solutions. All of this work is based on understanding quantities and the relationships between them.

In Mathematics I, students build on their prior experiences with data, developing more formal means of assessing how a model fits data. Students use regression techniques to describe approximately linear relationships between quantities. They use graphical representations and knowledge of the context to make judgments about the appropriateness of linear models. With linear models, they look at residuals to analyze the goodness of fit.

At previous grade levels, students were asked to draw triangles based on given measurements. They also gained experience with rigid motions (translations, reflections, and rotations) and developed notions about what it means for two objects to be congruent. In Mathematics I, students establish triangle congruence criteria based on analyses of rigid motions and physical constructions. They solve problems about triangles, quadrilaterals, and other polygons. They apply reasoning to complete geometric constructions and explain why the constructions work. Finally, building on their work with the Pythagorean Theorem in the grade-eight standards to find distances, students use a rectangular coordinate system to verify geometric relationships, including properties of special triangles and quadrilaterals and slopes of parallel and perpendicular lines.

Connecting Mathematical Practices and Content

The SMPs apply throughout each course and, together with the CA CCSSM, prescribe that students experience mathematics as a coherent, culturally relevant, and meaningful subject. The SMPs represent a picture of what it looks like for students to do mathematics and, to the extent possible, content instruction should include attention to appropriate practice standards.

The CA CCSSM call for an intense focus on the most critical material, allowing depth in learning, which is carried out through the SMPs. Connecting practices and content happens in the context of working on problems; the very first SMP is to make sense of problems and persevere in solving them. Figure A.11 gives examples of how students can engage in the SMPs in Mathematics I.

Figure A.11: Standards for Mathematical Practice—Explanation and Examples for Mathematics

Standards for Mathematical Practice	Explanation and Examples
SMP.1 Make sense of problems and persevere in solving them.	Students persevere when attempting to understand the differences between linear and exponential functions. They make diagrams of geometric problems to help make sense of the problems.
SMP.2 Reason abstractly and quantitatively.	Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.
SMP.3 Construct viable arguments and critique the reasoning of others. Students build proofs by induction and proofs by contradiction. CA 3.1 (for higher mathematics only).	Students reason through the solving of equations, recognizing that solving an equation involves more than simply following rote rules and steps. They use language such as "If …, then …" when explaining their solution methods and provide justification for their reasoning.
SMP.4 Model with mathematics.	Students apply their mathematical understanding of linear and exponential functions to many real-world problems, such as linear and exponential growth. Students also discover mathematics through experimentation and by examining patterns in data from real-world contexts.

Standards for Mathematical Practice	Explanation and Examples
SMP.5 Use appropriate tools strategically.	Students develop a general understanding of the graph of an equation or function as a representation of that object, and they use tools such as graphing calculators or graphing software to create graphs in more complex examples, understanding how to interpret the results.
SMP.6 Attend to precision.	Students use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure and labeling axes to clarify the correspondence with quantities in a problem.
SMP.7 Look for and make use of structure.	Students recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects.
SMP.8 Look for and express regularity in repeated reasoning.	Students see that the key feature of a line in the plane is an equal difference in outputs over equal intervals of inputs, and that the result of evaluating the expression $\frac{y_2 - y_1}{x_2 - x_1}$ for points on the line is always equal to a certain number m. Therefore, if (x, y) is a generic point on this line, the equation $m = \frac{y - y_1}{x - x_1}$ will give a general equation of that line.

SMP.4 holds a special place throughout the higher mathematics curriculum, as Modeling is considered its own conceptual category. Although the Modeling category does not include specific standards, the idea of using mathematics to model the world pervades all higher mathematics courses and should hold a significant place in instruction. Some standards are marked with a star (⋆) symbol to indicate that they are modeling standards—that is, they may be applied to real-world modeling situations more so than other standards.

Integrated Math II

The Mathematics II course focuses on quadratic expressions, equations, and functions and on comparing the characteristics and behavior of these expressions, equations, and functions to those of linear and exponential relationships from Mathematics I. The need for extending the set of rational numbers arises, and students are introduced to real and complex numbers. Links between probability and data are explored through conditional probability and counting methods and involve the use of probability and data in making and evaluating decisions.

The study of similarity leads to an understanding of right-triangle trigonometry and connects to quadratics through Pythagorean relationships. Circles, with their quadratic algebraic representations, finish out the course.

The courses in the Integrated Pathway follow the structure introduced in the kindergarten through grade eight levels of the CA CCSSM they present mathematics as a coherent subject and blend standards from different conceptual categories.

The standards in the integrated Mathematics II course come from the following conceptual categories: Modeling, Functions, Number and Quantity, Algebra, Geometry, and Statistics and Probability. The course content is explained below according to these conceptual categories, but teachers and administrators alike should note that the standards are not listed here in the order in which they should be taught. Moreover, the standards are not topics to be checked off after being covered in isolated units of instruction; rather, they provide content to be developed throughout the school year through rich instructional experiences.

What Students Learn in Mathematics II

In Mathematics II, students extend the laws of exponents to rational exponents and explore distinctions between rational and irrational numbers by considering their decimal representations. Students learn that when quadratic equations do not have real solutions, the number system can be extended so that solutions exist, analogous to the way in which extending whole numbers to negative numbers allows $x + 1 = 0$ to have a solution. Students explore relationships between number systems: whole numbers, integers, rational numbers, real numbers, and complex numbers. The guiding principle is that equations with no solutions in one number system may have solutions in a larger number system.

Students consider quadratic functions, comparing the key characteristics of quadratic functions to those of linear and exponential functions. They select from these functions to model phenomena. Students learn to anticipate the graph of a quadratic function by interpreting various forms of quadratic expressions. In particular, they identify the real solutions of a quadratic equation as the zeros of a related quadratic function. Students also learn that when quadratic equations do not have real solutions, the graph of the related quadratic function does not cross the horizontal axis. Additionally, students expand their experience with functions to include more specialized functions—absolute value, step, and other piecewise-defined functions.

Students in Mathematics II focus on the structure of expressions, writing equivalent expressions to clarify and reveal aspects of the quantities represented. Students create and solve equations, inequalities, and systems of equations involving exponential and quadratic expressions.

Building on probability concepts introduced in the middle grades, students use the language of set theory to expand their ability to compute and interpret theoretical and experimental probabilities for compound events, attending to mutually exclusive events, independent events, and conditional probability. Students use probability to make informed decisions, and they should make use of geometric probability models whenever possible.

Students apply their earlier experience with dilations and proportional reasoning to build a formal understanding of similarity. They identify criteria for similarity of triangles, use similarity to solve problems, and apply similarity in right triangles to understand right-triangle trigonometry, with particular attention to special right triangles and the Pythagorean Theorem. In Mathematics II, students develop facility with geometric proof. They use what they know about congruence and similarity to prove theorems involving lines, angles, triangles, and other polygons. They also explore a variety of formats for writing proofs.

In Mathematics II, students prove basic theorems about circles, chords, secants, tangents, and angle measures. In the Cartesian coordinate system, students use the distance formula to write the equation of a circle when given the radius and the coordinates of its center, and the equation of a parabola with a vertical axis when given an equation of its horizontal directrix and the coordinates of its focus. Given an equation of a circle, students draw the graph in the coordinate plane and apply techniques for solving quadratic equations to determine intersections between lines and circles, between lines and parabolas, and between two circles. Students develop informal arguments to justify common formulas for circumference, area, and volume of geometric objects, especially those related to circles.

Examples of Key Advances from Mathematics I

Students extend their previous work with linear and exponential expressions, equations, and systems of equations and inequalities to quadratic relationships.

- A parallel extension occurs from linear and exponential functions to quadratic functions: students begin to analyze functions in terms of transformations.
- Building on their work with transformations, students produce increasingly formal arguments about geometric relationships, particularly around notions of similarity.

Connecting Mathematical Practices and Content

The SMPs apply throughout each course and, together with the CA CCSSM, prescribe that students experience mathematics as a coherent, culturally relevant, and meaningful subject. The SMPs represent a picture of what it looks like for students to do mathematics and, to the extent possible, content instruction should include attention to appropriate practice standards.

The CA CCSSM call for an intense focus on the most critical material, allowing depth in learning, which is carried out through the SMPs. Connecting content and practices happens in the context of working on problems, as is evident in the first SMP (“Make sense of problems and persevere in solving them”). Figure A.12 offers examples of how students can engage in each mathematical practice in the Mathematics II course.

Figure A.12: Standards for Mathematical Practice—Explanation and Examples for Mathematics II

Standards for Mathematical Practice	Explanation and Examples
SMP.1 Make sense of problems and persevere in solving them.	Students persevere when attempting to understand the differences between quadratic functions and linear and exponential functions studied previously. They create diagrams of geometric problems to help make sense of the problems.
SMP.2 Reason abstractly and quantitatively.	Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.
SMP.3 Construct viable arguments and critique the reasoning of others. Students build proofs by induction and proofs by contradiction. CA 3.1 (for higher mathematics only).	Students construct proofs of geometric theorems based on congruence criteria of triangles. They understand and explain the definition of *radian measure.*
SMP.4 Model with mathematics.	Students apply their mathematical understanding of quadratic functions to real-world problems. Students also discover mathematics through experimentation and by examining patterns in data from real-world contexts.
SMP.5 Use appropriate tools strategically.	Students develop a general understanding of the graph of an equation or function as a representation of that object, and they use tools such as graphing calculators or graphing software to create graphs in more complex examples, understanding how to interpret the result.
SMP.6 Attend to precision.	Students begin to understand that a *rational number* has a specific definition and that *irrational numbers* exist. When deciding if an equation can describe a function, students make use of the definition of *function* by asking, “Does every input value have exactly one output value?”
SMP.7 Look for and make use of structure.	Students apply the distributive property to develop formulas such as $(a \pm b)^2 = a^2 \pm 2an + b^2$. They see that the expression $5 + (x - 2)^2$ takes the form of “5 plus ‘something’ squared,” and therefore that expression can be no smaller than 5.

Standards for Mathematical Practice	Explanation and Examples
SMP.8 Look for and express regularity in repeated reasoning.	Students notice that consecutive numbers in the sequence of squares 1, 4, 9, 16, and 25 always differ by an odd number. They use polynomials to represent this interesting finding by expressing it as $(n+1)^2-n^2=2n+1$.

SMP.4 holds a special place throughout the higher mathematics curriculum, as Modeling is considered its own conceptual category. Although the Modeling category does not include specific standards, the idea of using mathematics to model the world pervades all higher mathematics courses and should hold a significant place in instruction. Some standards are marked with a star (⋆) symbol to indicate that they are modeling standards—that is, they may be applied to real-world modeling situations more so than other standards. Modeling in higher mathematics centers on problems that arise in everyday life, society, and the workplace. Such problems may draw upon mathematical content knowledge and skills articulated in the standards prior to or during the Mathematics II course.

Integrated Math III

In the Mathematics III course, students expand their repertoire of functions to include polynomial, rational, and radical functions. They also expand their study of right-triangle trigonometry to include general triangles. And, finally, students bring together all of their experience with functions and geometry to create models and solve contextual problems. The courses in the Integrated Pathway follow the structure introduced in the kindergarten through grade eight levels of the CA CCSSM; they present mathematics as a coherent subject and blend standards from different conceptual categories.

The standards in the integrated Mathematics III course come from the following conceptual categories: Modeling, Functions, Number and Quantity, Algebra, Geometry, and Statistics and Probability. The course content is explained below according to these conceptual categories, but teachers and administrators alike should note that the standards are not listed here in the order in which they should be taught. Moreover, the standards are not topics to be checked off after being covered in isolated units of instruction; rather, they provide content to be developed throughout the school year through rich instructional experiences.

What Students Learn in Mathematics III

In Mathematics III, students understand the structural similarities between the system of polynomials and the system of integers. Students draw on analogies between polynomial arithmetic and base-ten computation, focusing on properties of operations, particularly the distributive property. They connect multiplication of polynomials with multiplication of multi-digit integers and division of polynomials with long division of integers. Students identify zeros of polynomials and make connections between zeros of polynomials and solutions of polynomial equations. Their work on polynomial expressions culminates with the Fundamental Theorem of Algebra. Rational numbers extend the arithmetic of integers by allowing division by all numbers except 0. Similarly, rational expressions extend the arithmetic of polynomials by allowing division by all polynomials except the zero polynomial. A central theme of working with rational expressions is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers.

Students synthesize and generalize what they have learned about a variety of function families. They extend their work with exponential functions to include solving exponential equations with logarithms. They explore the effects of transformations on graphs of diverse functions, including functions arising in an application, in order to abstract the general principle that transformations on a graph always have the same effect, regardless of the type of the underlying functions.

Students develop the Laws of Sines and Cosines in order to find missing measures of general (not necessarily right) triangles. They are able to distinguish whether three given measures (angles or sides) define 0, 1, 2, or infinitely many triangles. This discussion of general triangles opens up the idea of trigonometry applied beyond the right triangle—that is, at least to obtuse angles. Students build on this idea to develop the notion of radian measure for angles and extend the domain of the trigonometric functions to all real numbers. They apply this knowledge to model simple periodic phenomena.

Students see how the visual displays and summary statistics they learned in previous grade levels or courses relate to different types of data and to probability distributions. They identify different ways of collecting data—including sample surveys, experiments, and simulations—and recognize the role that randomness and careful design play in the conclusions that may be drawn.

Finally, students in Mathematics III extend their understanding of modeling: they identify appropriate types of functions to model a situation, adjust parameters to improve the model, and compare models by analyzing appropriateness of fit and by making judgments about the domain over which a model is a good fit. The description of modeling as “the process of choosing and using mathematics and statistics to analyze empirical situations, to understand them better, and to make decisions” (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA/CCSSO], 2010) is one of the main themes of this course. The discussion about modeling and the diagram of the modeling cycle that appear in this chapter should be considered when students apply knowledge of functions, statistics, and geometry in a modeling context.

Examples of Key Advances from Mathematics II

- Students begin to see polynomials as a system analogous to the integers that they can add, subtract, multiply, and so forth. Subsequently, polynomials can be extended to rational expressions, which are analogous to rational numbers.
- Students extend their knowledge of linear, exponential, and quadratic functions to include a much broader range of classes of functions.
- Students begin to examine the role of randomization in statistical design.

Connecting Mathematical Practices and Content

The SMPs apply throughout each course and, together with the CA CCSSM, prescribe that students experience mathematics as a coherent, culturally relevant, and meaningful subject. The SMPs represent a picture of what it looks like for students to do mathematics and, to the extent possible, content instruction should include attention to appropriate practice standards. The Mathematics III course offers ample opportunities for students to engage with each SMP; figure A.13 offers some examples.

Figure A.13: Standards for Mathematical Practice—Explanation and Examples for Mathematics III

Standards for Mathematical Practice	Explanation and Examples
SMP.1 Make sense of problems and persevere in solving them.	Students apply their understanding of various functions to real-world problems. They approach complex mathematics problems and break them down into smaller problems, synthesizing the results when presenting solutions.
SMP.2 Reason abstractly and quantitatively.	Students deepen their understanding of variables—for example, by understanding that changing the values of the parameters in the expression has consequences for the graph of the function. They interpret these parameters in a real-world context.
SMP.3 Construct viable arguments and critique the reasoning of others. Students build proofs by induction and proofs by contradiction. CA 3.1 (for higher mathematics only).	Students continue to reason through the solution of an equation and justify their reasoning to their peers. Students defend their choice of a function when modeling a real-world situation.
SMP.4 Model with mathematics.	Students apply their new mathematical understanding to real-world problems, making use of their expanding repertoire of functions in modeling. Students also discover mathematics through experimentation and by examining patterns in data from real-world contexts.
SMP.5 Use appropriate tools strategically.	Students continue to use graphing technology to deepen their understanding of the behavior of polynomial, rational, square root, and trigonometric functions.
SMP.6 Attend to precision.	Students make note of the precise definition of *complex number,* understanding that real numbers are a subset of complex numbers. They pay attention to units in real-world problems and use unit analysis as a method for verifying their answers.
SMP.7 Look for and make use of structure.	Students understand polynomials and rational numbers as sets of mathematical objects that have particular operations and properties. They understand the periodicity of sine and cosine and use these functions to model periodic phenomena.

Standards for Mathematical Practice	Explanation and Examples
SMP.8 Look for and express regularity in repeated reasoning.	Students observe patterns in geometric sums—for example, that the first several sums of the form $\sum_{k=0}^{n} 2^k$ can be written as follows: $1 = 2^1 - 1$ $1 + 2 = 2^2 - 1$ $1 + 2 + 4 = 2^3 - 1$ $1 + 2 + 4 + 8 = 2^4 - 1$ Students use this observation to make a conjecture about any such sum.

Mathematics: Investigating and Connecting Pathway

The Mathematics: Investigating and Connecting pathway outline given here should be viewed as a next iteration of the Integrated Mathematics courses outlined in the previous section. The courses *Mathematics: Investigating and Connecting 1*, *Mathematics: Investigating and Connecting 2*, and *Mathematics: Investigating and Connecting 3* (MIC 1, 2, 3) are *implementations* of the Math I, Math II, and Math III sample content outlines in the California Common Core State Standards for Mathematics (CA CCSSM) (augmented by some data clusters which are moved from Integrated Math III into MIC 1 and MIC 2).

Following the common experience of MIC 1 and MIC 2, the MIC 3 course outline includes options for curriculum designers and districts to build versions that emphasize different types of investigations to situate student activities, and perhaps distribute student effort differently between the various Conceptual Categories of the CA CCSSM. This gives districts opportunities to respond to recent policy guidance (Daro and Asturias, 2019) suggesting that students have choices in their mathematics pathways following the first two common higher mathematics (high school) courses. All MIC 3 courses should continue the MIC 1 and 2 emphasis on developing mathematical understanding in response to students' authentic questions; and should offer a path to multiple twelfth-grade courses (including University of California [UC] A–G courses), so that students are not locked into a track with their MIC third year choice.

The Big Ideas for MIC 1 and 2 are described next, followed by an in-depth look at each of the four Content Connections across the MIC courses. The CCs are illustrated with a relevant vignette and with CA CCSSM content domains listed for each. See the CA CCSSM for the full language of standards in the domain. Note that almost all tasks and investigations will involve multiple domains, with a goal of building connections across multiple mathematical ideas.

MIC Pathway Big Ideas

The state of California set out the most important mathematical content and practices by highlighting a collection of big ideas in mathematics, TK-10 in the Digital Learning and Standards Initiative (CDE, 2021). In this document, the CACCSSM content standards and Standards for Mathematical Practice in grades TK-10 were organized into a set of Big Ideas, which themselves are organized into the Content Connections.

Figure A.14 presents the progression of Big Ideas for the MIC 1 and 2 course sequence. The network maps, in Figures A.15 and A.17, highlight important and foundational content, shown as nodes, for each grade level. As students explore and investigate with the Big Ideas, they will likely encounter many different content standards and note the connections between them. The size of a node relates to the number of connections it has with other Big Ideas. The connections between Big Ideas are made when the two connected Big Ideas contain one or more of the same standards.

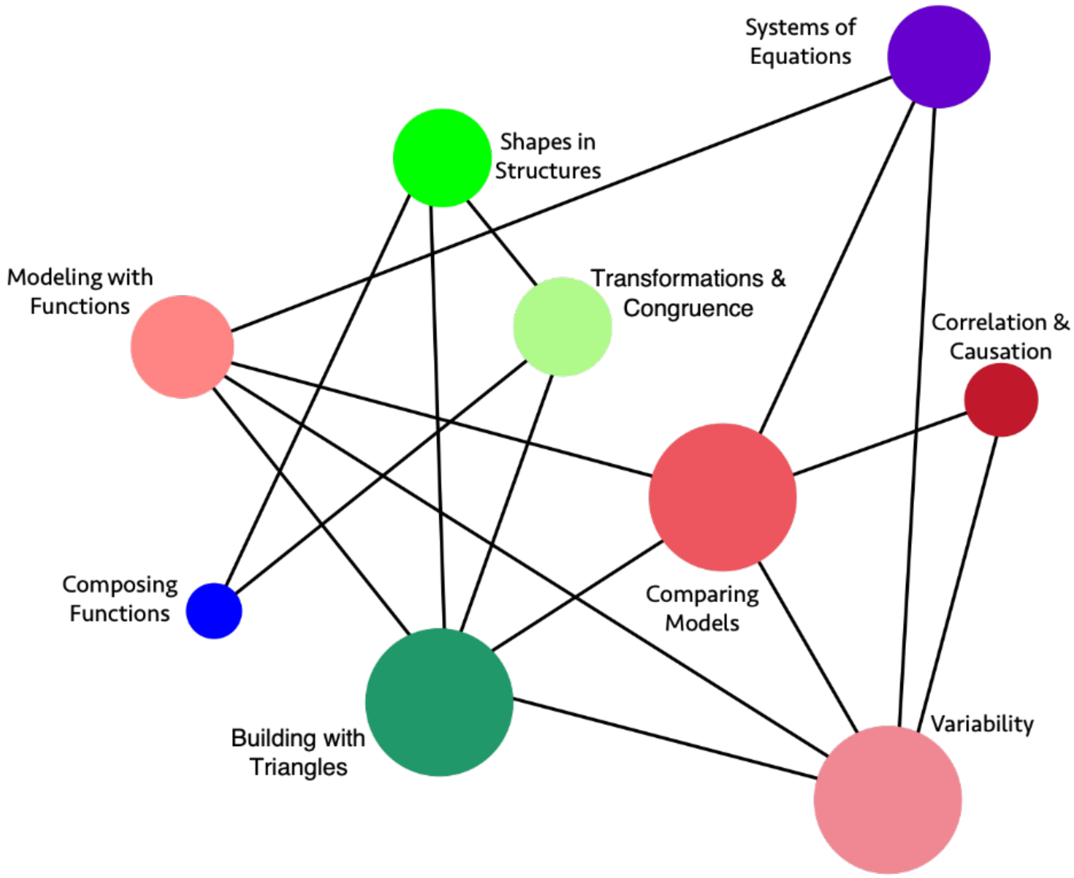
The colors in the nodes correspond to the Content Connections, Big Ideas, and Standards tables, Tables A.16 and A.18, which follow each of the network diagrams for the two courses. The Big Ideas (middle column) are situated within their broader Content Connection (left column), and the CACCSSM content standards (right column) which can be addressed for each Big Idea are indicated.

Figure A.14: A Progression Chart of Big Ideas through MIC 1 and 2

Content Connections	Big Ideas: MIC 1	Big Ideas: MIC 2
Communicating Stories with Data	Modeling with functions	The shape of distributions

Content Connections	Big Ideas: MIC 1	Big Ideas: MIC 2
Communicating Stories with Data	Comparing models	Geospatial data
Communicating Stories with Data	Variability	Probability modeling
Communicating Stories with Data	Correlation & causation	Experimental models and functions
Exploring Changing Quantities	Modeling with functions	The shape of distributions
Exploring Changing Quantities	Comparing models	Equations to predict & model
Exploring Changing Quantities	Variability	Experimental models & functions
Exploring Changing Quantities	Systems of equations	Transformation & similarity
Taking Wholes Apart, Putting Parts Together	Systems of equations	Functions in the world
Taking Wholes Apart, Putting Parts Together	Composing functions	Polynomial identities
Taking Wholes Apart, Putting Parts Together	Shapes in structures	Function representations
Taking Wholes Apart, Putting Parts Together	Building with triangles	n/a
Discovering shape and space	Shapes in structures	Circle relationships
Discovering shape and space	Building with triangles	Trig functions
Discovering shape and space	Transformations & congruence	Transformation & similarity

Figure A.15: High School MIC 1 Big Ideas



[Link to long description]

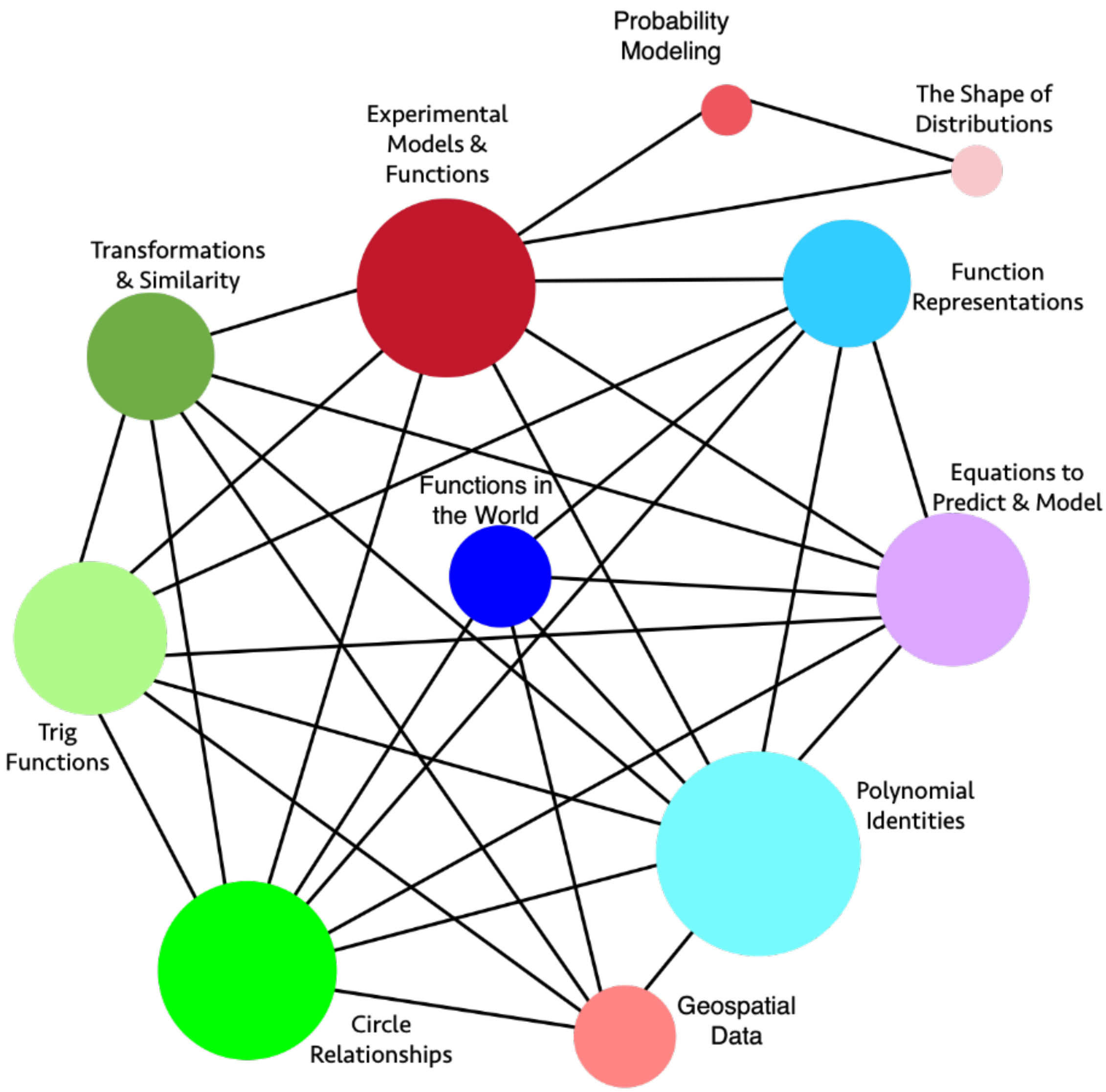
Figure A.16: High School MIC 1 Content Connections, Big Ideas, and Standards

Content Connection	Big Idea	Integrated 1 Standards
Communicating Stories with Data & Exploring Changing Quantities	**Modeling with Functions**	**N-Q.1, N-Q.2, N-Q.3, A-CED.2, F-BF.1, F-IF.1, F-IF.2, F-IF.4, F-LE.5, S-ID.7, A-CED.1, A-CED.2, A-CED.3, A-SSE.1:** Build functions that model relationships between two quantities, including examples with inequalities; using units and different representations. Describe and interpret the relationships modeled using visuals, tables, and graphs.

Content Connection	Big Idea	Integrated 1 Standards
Communicating Stories with Data & Exploring Changing Quantities	**Comparing Models**	**F-LE.1, F-LE.2, F-LE.3, F-IF.4, F-BF.1, F-LE.5, S-ID.7, S-ID.8, A-CED.1, A-CED.2, A-CED.3, A-SSE.1:** Construct, interpret, and compare linear, quadratic, and exponential models of real data, and use them to describe and interpret the relationships between two variables, including inequalities. Interpret the slope and constant terms of linear models, and use technology to compute and interpret the correlation coefficient of a linear fit.
Communicating Stories with Data & Exploring Changing Quantities	**Variability**	**S-ID.5, S-ID.6, S-ID.7, S-ID.1, S-ID.2, S-ID.3, S-ID.4, A-SSE.1:** Summarize, represent, and interpret data. For quantitative data, use a scatter plot and describe how the variables are related. Summarize categorical data in two-way frequency tables and interpret the relative frequencies.
Communicating Stories with Data	**Correlation & Causation**	**S-ID.9, S-ID.8, S-ID.7:** Explore data that highlights the difference between correlation and causation. Understand and use correlation coefficients, where appropriate. (see resource section for classroom examples).
Exploring Changing Quantities & Taking Wholes Apart, Putting Parts Together	**Systems of Equations**	**A-REI.1, A-REI.3, A-REI.4, A-REI.5, A-REI.6, A-REI.7, A-REI.10, A-REI.11, A-REI.12, NQ.1, A-SEE.1:** Students investigate real situations that include data for which systems of 1 or 2 equations or inequalities are helpful, paying attention to units. Investigations include linear, quadratic, and absolute value. Students use technology tools strategically to find their solutions and approximate solutions, constructing viable arguments, interpreting the meaning of the results, and communicating them in multidimensional ways.
Taking Wholes Apart, Putting Parts Together	**Composing Functions**	**F-BF.3, F-BF.2, F-IF.3:** Build and explore new functions that are made from existing functions, and explore graphs of the related functions using technology. Recognize sequences are functions and are defined recursively.

Content Connection	Big Idea	Integrated 1 Standards
Taking Wholes Apart, Putting Parts Together & Discovering Shape and Space	**Shapes in Structures**	**G-CO.6, C-CO.7, C-CO.8, G-GPE.4, G-GPE.5, G.GPE.7, F.BF.3:** Perform investigations that involve building triangles and quadrilaterals, considering how the rigidity of triangles and non-rigidity of quadrilaterals influences the design of structures and devices. Study the changes in coordinates and express the changes algebraically.
Taking Wholes Apart, Putting Parts Together & Discovering Shape and Space	**Building with Triangles**	**G-GPE.4, G-GPE.5, G-GPE.6, GPE.7, F-LE.1, F-LE.2, A-CED.2:** Investigate with geometric figures, constructing figures in the plane, relating the distance formula to the Pythagorean Theorem, noticing how areas and perimeters of polygons change as the coordinates change. Build with triangles and quadrilaterals, noticing positions and movement, and creating equations that model the changing edges using technology.
Discovering Shape and Space	**Transformations & Congruence**	**G-CO.1, G-CO.2, G-CO.3, G-CO.4, G-CO.5, G-CO.12, G-CO.13, G-GPE.4, G-GPE.5, G.GPE.7, F-BF.3:** Explore congruence of triangles, including quadrilaterals built from triangles, through geometric constructions. Investigate transformations in the plane. Use geometry software to study transformations, developing definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, and parallel lines. Express translations algebraically.

Figure A.17: Big Ideas Map for MIC 2



[Link to long description](#)

Figure A.18: High School MIC 2 Content Connections, Big Ideas, and Standards

Content Connection	Big Idea	Integrated 2 Standards
Communicating Stories with Data	**Probability Modeling**	**S.CP.1, S.CP.2, S.CP.3, S.CP.4, S.CP.5, S-IC.1, S-IC.2, S-IC.3, S.MD.6, S.MD.7:** Explore and compare independent and conditional probabilities, interpreting the output in terms of the model. Construct and interpret two-way frequency tables of data as a sample space to determine if the events are independent, and use the data to approximate conditional probabilities. Examples of topics include product and medical testing, and player statistics in sports.
Communicating Stories with Data	**The shape of distributions**	**S-IC.1, S-IC.2, S-IC.3, S-ID.1, S-ID.2, S-ID.3, S-MD.1, S-MD.2:** Consider the shape of data distributions to decide on ways to compare the center and spread of data. Use simulation models to generate data, and decide if the model produces consistent results.
Communicating Stories with Data & Exploring Changing Quantities	**Experimental Models & Functions**	**S-ID.1, S-ID.2, S-ID.3, S- ID.6, S-ID.7, S-IC.1, S-IC.2, S-IC.3, A-CED.1, A-REI.1, A-REI.4, F-IF.2, F-IF.3, F-IF.4, F-BF.1, F-LE.1, F-TF.2, A-APR.1:** Conduct surveys, experiments, and observational studies - drawing conclusions and making inferences. Compare different data sources and what may be most appropriate for the situation. Create and interpret functions that describe the relationships, interpreting slope and the constant term when linear models are used. Include quadratic and exponential models when appropriate, and understand the meaning of outliers.
Communicating Stories with Data	**Geospatial Data**	**G-MG.1, G-MG.2, G-MG.3, F-LE.6, G-GPE.4, G-GPE.6, G-SRT.5, G-CO.1, G-CO.2, G-CO.12, G-C.2, G-C.5:** Explore geospatial data that represent either locations (e.g., maps) or objects (e.g., patterns of people's faces, road objects for driverless cars) and connect to geometric equations and properties of common shapes. Demonstrate how a computer can measure the distance between two points using geometry and then account for constraints (e.g., distance and then roads for directions) and multiple points with triangulation. Model what shapes and geometric relationships are most appropriate for different situations.

Content Connection	Big Idea	Integrated 2 Standards
Exploring Changing Quantities	**Equations to Predict & Model**	**A-CED.1, A-CED.2, A-REI.4, A-REI.1, A-REI.2, A-REI.3, F.IF.4, F.IF.5, F.IF.6, F.BF.1, F.BF.3, A-APR.1:** Model relationships that include creating equations or inequalities, including linear, quadratic, and absolute value. Use the equations or inequalities to make sense of the world or to make predictions, understanding that solving equations is a process of reasoning. Make sense of the real situation, using multiple representations, such as graphs, tables, and equations.
Taking Wholes Apart, Putting Parts Together	**Functions in the World**	**F-LE.3, F-LE.6, F-IF.9, N-RN.1, N-RN.2, A-SSE.1, A-SSE.2:** Apply quadratic functions to the physical world, such as motion of an object under the force of gravity. Produce equivalent forms of the functions to reveal zeros, max and min, and intercepts. Investigate how functions increase and decrease, and compare the rates of increase or decrease to linear and exponential functions.
Taking Wholes Apart, Putting Parts Together	**Polynomial Identities**	**A-SSE.1, A-SSE.2, A-APR.1, A-APR.3, A-APR.4, G-GMD.2, G-MG.1, S-IC.1, S-MD.2:** Prove polynomial identities, and use them to describe numerical relationships, using a computer algebra system to rewrite polynomials. Use the binomial theorem to solve problems, appreciating the connections with Pascal's triangle.
Taking Wholes Apart, Putting Parts Together	**Functions Representations**	**F-IF.4, F-IF.5, F-IF.6, F-IF.7, F-IF.8, F-IF.9, N-RN.1, N-RN.2, F-LE.3, A-APR.1:** Interpret functions representing real world applications in terms of the data understanding key features of graphs, tables, domain, and range. Compare properties of two functions each represented in different ways (algebraically, graphically, numerically, in tables or by written/verbal descriptions).

Content Connection	Big Idea	Integrated 2 Standards
Discovering Shape and Space & Exploring Changing Quantities	**Transformations & Similarity**	**G-SRT.1, G- SRT.2, G-SRT.3, , A-CED.2, G-GPE.4, F-BF.3, F-IF.4, A-APR.1:** Explore similarity and congruence in terms of transformations, noticing the changes dilations have on figures and the effect of scale factors. Discover how coordinates can be used to describe translations, rotations, and reflections, and generalize findings to model the transformations using algebra.
Discovering Shape and Space	**Circle Relationships**	**G-C.1, G-C.2, G-C.3, G-C.4, G-C.5, G-GPE.1, A-REI.7, A-APR.1, F-IF.9:** Investigate the relationships of angles, radii, and chords in circles, including triangles and quadrilaterals that are inscribed and circumscribed. Explore arc lengths and areas of sectors using the coordinate plane. Relate the Pythagorean Theorem to the equation of the circle given the center and radius, and solve simple systems where a line intersects the circle.
Discovering Shape and Space	**Trig Functions**	**G-TF.2, G-GPE.1, G-GMD.2, G-MG.1, A-APR.1:** Model periodic phenomena with trigonometric functions. Translate between geometric descriptions and the equation for a conic section. Visualize relationships between 2-D and 3-D objects.

MIC Pathway Outline: The Content Connections

In this section, the development of mathematical content through the MIC courses is organized according to the Content Connections, in order to keep a focus on big ideas in designing instruction. The MIC courses are implementations of the Integrated I, II, and III Model Course Outlines in the CA CCSSM, so the content learning expectations can also be reviewed in those Model Course Outlines. Designers of instructional materials built to enact the MIC outline will need to pay careful attention to the Model Course Outlines in the CA CCSSM.

Course Progression of CC 1: Communicating Stories with Data

The Mathematics: Investigating and Connecting pathway gives prominence to reasoning about and with data, reflecting the growing importance of data as the source of most mathematical situations that students will encounter in their lives. Content Connection 1 is the only Content Connection in which standards differ from those in the CA CCSSM Integrated Mathematics model course outlines. Given the rapidly increasing importance of data literacy, many Statistics and Probability standards that are in year three of the model course outlines are here addressed through all years of the MIC pathway.

The progression of data literacy is addressed in more detail in Chapter 5. Briefly, in MIC 1, students should experience repeated random processes and keep track of the outcomes, to begin to develop a sense of the likelihood of certain types of events. They must have experience generating authentic questions that data might help to answer (investigative questions), and should have opportunities to gather some data to attempt to answer their questions. They should plot data on scatter plots, and informally fit linear and exponential functions when data appear in the plot to demonstrate a relationship (using physical objects like spaghetti or pipe cleaners, or online graphing technology).

In MIC 2, investigations should be designed to build students' understanding of probability as the basis for statistical claims. For functions modeling relationships between quantities, "strength of fit" is introduced (informally at first by comparing weak and strong associations with identical linear models) as a measure of how much of the observed variability is explained by the model; it measures predictive ability of the model.

MIC 3 has many student investigations driven by data. Students generate questions, design data collection, search for available existing data, analyze data, and represent data and results of analysis. They use powerful technological tools to help with all of these tasks. Much of the content in all Content Connections is situated in stories told through data. Other MIC 3 investigations, however, will be based on a structural understanding of the context: A function to represent the height at time t seconds of a ball thrown at a given upward velocity; a model to represent the total cost of ownership of a car over n years based on sales price, fuel costs, and average maintenance costs. For these investigations, data may play a bigger role in the parameter-estimation and validation stages of modeling (see below in Content Connection 2).

To respond to varying student interests, distinct MIC 3 courses can be designed by emphasizing data-driven investigations or more structural modeling, and/or by situating student work in different contexts (for example, observable physical, environmental, and mathematical phenomena; or social and economic community data).

CC1: Communicating Stories with Data

MIC 1 Key Ideas	MIC 2 Key Ideas	MIC 3 Key Ideas
• Random processes • Sense of likelihood • Generating questions • Gather data • Scatter plots • Informally fit linear and exponential	• Understanding that statistical claims are based on probability • Beginning understanding of strength of fit • Continued generation of questions: investigative, interrogative, data gathering • Additional representations of data	• Investigations driven by data • Generate investigative questions • Design data collection or find existing data • Analyze data • Represent and report conclusions • Use probability to make decisions • Use sophisticated technological tools

Course Progression of CC 2: Exploring Changing Quantities

Investigations that develop the mathematical content of Content Connection 2: Exploring Changing Quantities should span the range of the Drivers of Investigation, with particular attention paid to culturally relevant activities in DI 2 (Predict What Could Happen) and DI 3 (Impact the Future), since these types of activities most easily help students experience mathematics as a useful lens for their lives. This Content Connection includes much of the content of the CA CCSSM Conceptual Categories *Functions, Modeling,* and *Algebra.* Also note that many investigations in Content Connection 1: *Telling Stories with Data* will involve extensive work with Content Connection 2 content.

In MIC 1, tasks and explorations in this Content Connection should focus mostly on quantities that change with respect to time or “step number.” Relationships should be primarily linear and exponential, with some other relationships explored only informally (for example, predicting using a plot of known points and a pipe cleaner for interpolating or extrapolating). Most questions begin with “When will…?” or “At this time, what will…?” Students must generate many of the questions for exploration, and even some of the contexts for questioning. For example, “What are some things that affect your life, that change over the course of the school year?” can generate contexts to explore.

In MIC 1, quantities should include linear measurement (length and distance), population growth (e.g., bacteria), and interest (both deposits and debts), among many other contexts that generate linear and exponential growth. Typically, students will approach these situations recursively at first, seeing either a constant additive (linear growth: same amount added each time period) or constant multiplicative (exponential growth: quantity grows by the same factor or percent each time period). Most of the mathematical work emerges from attempts to find or predict the value of the changing quantity at a point in the future or at a point in between known values; then to express the value of the quantity at an arbitrary point in time. Verbal, graphical, and symbolic representations should all appear as appropriate, with emphasis on the connections between them and the features of the relationship between quantities that each representation helps to make clear.

Beginning in MIC 1 and continuing through MIC 2, the general notion of function should be developed and synthesized through this Content Connection, typically building from different situations that generate the same linear or exponential relationship, then noting the similarities, and discussing function notation as a way to capture multiple situations at once. (See the discussion of abstraction in the “Rigor” section in Chapter 1). Problems framed in terms of abstract functions (that is, functions given as formulas, graphs, or tables without an accompanying context) should frequently include prompts to “invent a context that this function (or equation or expression) might represent.” This process is different from mathematical modeling, but it is helpful in maintaining the connection between mathematics and students’ lives that is so important in order for students to see mathematics as having value.

In MIC 2, measured and observed quantities that change relative to other quantities besides time or step number should be investigated, in addition to the time/step relationships in MIC 1. Relationships modeled should expand to include quadratic and logarithmic, in addition to linear and exponential relationships explored in MIC 1. The general idea of function should be further developed as an abstraction of repeated efforts to understand, describe, and use relationships in particular contexts.

In MIC 3, one focus is the creation of function models for relationships that are observed through data, and the use and interpretation of those models. At first, these models should be guided by student-generated ad-hoc methods, such as:

- We used a yardstick on the graph and moved it around until it was as close as possible to all the dots.
- We measured the distance the car went when we raised the high end of the ramp to different heights. When we graphed it, it looked sort of like a line going up. On average, raising the ramp by one inch increased the car's distance by three and 1/4 inches, so we decided to try 3.25 as the slope for our line.
- We used Desmos to graph the area for different scale factors, and it curved upward. We first tried graphing exponential functions to see if they would match up, but none of them looked right. Then we tried quadratic functions and just played around with the numbers until they looked right with our dots.

Such ad-hoc methods should lead to discussions about what makes one proposed function "fit" the data better than another, and activities and should develop a conceptual idea (not by-hand computational skill) that the "best fit" function minimizes the total distance of all the data points from the function—while pointing out that it is actually *vertical* distances that are minimized, and that most software systems minimize the sum of the *squared* vertical distances, not the sum of the (absolute) vertical distances.

Later, students use appropriate technological tools to generate "best fit" functions, and use those functions as models for the relationships, in order to predict one quantity given the other. Extrapolating beyond known data should be contrasted with interpolating within.

In MIC 3, other functional models will be driven primarily by understood or theorized underlying structure governing the relationship between quantities, rather than by data about the relationship. For instance, the notion that speed of a vehicle changes at a constant rate if a constant force is applied is consistent with many students' experience (within a reasonable range and with some important simplifying assumptions!). Given this, a relationship between time and distance traveled can be developed and used to answer questions about the context. Data points can then be used to select the parameters (constants) of the model. (The mathematics of this example has been used in one of California's longest court cases over a speeding ticket; see Moore, 2009).

In all courses, investigations should include situations requiring solving equations and systems of equations. Such questions as these will necessitate such solutions:

- When will one quantity reach a fixed value?
- When will two different quantities that change over time be equal?
- When will one be greater than the other?
- At a fixed time, what is the rank order of the quantities?
- What value of (one quantity) corresponds to (a) specified value(s) of (other quantity[ies])?

Content Connection 2: *Exploring changing quantities* includes much of the content of the CA CCSSM Conceptual Categories *Functions, Modeling,* and *Algebra.* Modeling and Algebra are also heavily represented in Content Connection 3: Taking Wholes Apart, Putting Parts Together. In addition to these three, Content Connection 2 includes some CACCSSM domains from other Conceptual categories. Also note that many investigations in Content Connection 1: *Telling Stories with Data* will involve extensive work in Content Connection 2 content.

CC2: Exploring Changing Quantities

MIC 1 Key Ideas	MIC 2 Key Ideas	MIC 3 Key Ideas
• Changing quantities over time • Primarily linear and exponential • “When will…?” and “At what time will…?” questions • Student-generated questions • Verbal, graphical, numerical, symbolic representations • Emphasis on connections between representations • General notion of function as way to represent multiple situations	• Change with respect to time and other variables • Quadratic and logarithmic in addition to linear and exponential • More complex questions: rate and accumulation • Student-generated questions • Verbal, graphical, numerical, symbolic representations • Continued development of general notion of function	• Functions that model relationships observed in data • Models built by representing underlying structure • Interpretation and use of models to predict and make decisions • Build models informally at first • Technology handles computation • Notion of strength of fit: how much of variation is explained by model • Extrapolation vs interpolation • Multiple relationships modeled, requiring solving systems of equations to answer some questions

Course Progression of CC 3: Taking Wholes Apart, Putting Parts Together

This is the most unfamiliar of the Content Connections, focusing as it does on content in mathematics where the assembly of simple parts into a more complex understanding (and/or the corresponding disassembly) *is* the bulk of the intellectual work. A good example at the high school level is the functions exploration described in Chapter 4. Examples abound in geometry (also in CC 4) and modeling (also in CC 2).

In MIC 1, students interpret the structure of expressions by connecting parts of an expression (terms, factors, coefficients) with their meaning in the given context (primarily in linear expressions and in exponential expressions with integer exponents). They build new functions from existing ones—for instance, a constant term plus a proportional term, or a constant multiple of $f(x) = x^3$—and examine the effect of these combinations of known functions, and the meaning of these effects in terms of the quantities represented. In plane geometry, they experiment to see that, and then demonstrate why, a combination (composition) of rigid transformations is another rigid transformation, and build up rigid motions as compositions in order to demonstrate congruence of different figures. Steps in geometric constructions are understood as ways to build additional structure that can be used to produce a desired result (such as a copy of a segment or angle, or an equilateral triangle).

MIC 2 uses Content Connection 3 investigations to explore properties of the real numbers as ways in which real numbers can be combined, and to extend these properties to new numbers (e.g., extending properties of exponents to rational exponents). Investigating the structure of expressions by understanding the contributions of different parts to the whole expression continues from MIC 1. Equivalent expressions, and arithmetic with polynomials and rational expressions, are explored as different ways to put parts together, in order to highlight different features. Composing functions is a new way to build new functions from old, and frames the exploration of graph transformations such as replacing $f(x)$ by $f(kx)$, $kf(x)$, or $f(x + k)$ for specific values of k. Finally, explorations of probabilistic events made up of smaller events drives the ideas of independence and conditional probability.

In MIC 3's data-driven investigations, students begin by searching for, gathering, or examining data about their authentic questions, with the aim of exploring the effects of one or more quantity(ies) on another quantity of interest, and exploring the way that those effects combine. Functional models developed to represent relationships between quantities may have parts (such as terms, factors, coefficients) corresponding to different aspects influencing the quantity of interest. Thus, understanding the structure of polynomial and rational functions is a means to explaining observed relationships, and writing equivalent expressions helps to explain different characteristics of those observed relationships. Geometric measurement and dimension, and modeling with geometry, serve to build models of systems that generate the data being explored. For example, gathering data on leaf surface area of a species of plant as a function of some linear measurement (e.g., height or stem/trunk diameter), and then attempting to use that data to estimate leaf surface area for a larger specimen, will require that students wrestle with questions of dimension (does leaf surface area grow more like the surface area of the trunk or like the volume of the trunk?).

In MIC 3 modeling investigastions, students may investigate features of quadratic functions that lead to two real zeros, one real zero, and no real zeros; the latter leads to complex roots and a demonstration of the Fundamental Theorem of Algebra for quadratics, as well as to understanding the relationship between zeros and factors of polynomials. Polynomials up to degree 3 can be developed to meet building design challenges involving scaling (How much paint? How much trim? What capacity is needed for the heating system?), emphasizing the meaning in context of each term. Similarly, phenomena that exhibit periodic behavior (outside temperature over time, for instance) can be modeled by assembling terms (of the form $a \sin(b(x - c))$) representing different influences (daily cycle and seasonal cycle, for instance).

Content Connection 3: *Taking Wholes Apart, Putting Parts Together* includes parts of the content of the CA CCSSM Conceptual Categories of *Algebra, Modeling, Geometry, and Functions.* Modeling is also heavily represented in Content Connection 2: *Exploring Changing Quantities*, and Geometry is the content of Content Connection 4: *Discovering Shape and Space*.

CC3: Taking Wholes Apart, Putting Parts Together

MIC 1 Key Ideas	MIC 2 Key Ideas	MIC 3 Key Ideas
• Connect parts of expressions to meaning in context • Build new functions from existing • Build new rigid motions from old • View steps of geometric construction as building structure	• Structure of expressions via contributions of different parts • Equivalent expressions as different ways to put parts together • Properties of real numbers as tools for combining • Composing functions as a way to build new functions from old • Compound probability events	• Data science itself is a way to assemble a coherent picture by assembling many individual observations • Modeling is a process of creating mathematical representations of different aspects of a system • Building models from data or structure by combining different effects or terms for observed patterns or different aspects of structure • Geometric measurement and dimension to understand features of observed data

Course Progression of CC 4: Discovering Shape and Space

In grades three through five, students develop many foundational notions of two- and three-dimensional geometry, such as area (including surface area of three-dimensional figures), perimeter, angle measure, and volume.

Shape and space work in grades six through eight includes (Common Core Standards Writing Team, 2016):

- In plane geometry: area via decomposition, relationships between geometric figures and drawing shapes with specified conditions, and congruence and similarity using physical models, transparencies, and software;
- In geometric measurement: Solving real-world and mathematical problems involving area, surface area, volume, angle measure, and volume; drawing and constructing geometric figures and describing the relationships between them;

- In analytic geometry (connecting geometry and algebra): plotting points in the plane and graphing relationships between quantities.

For a more detailed description of the content in progression, see the Geometry, 7–8, High School progression (Common Core State Standards Writing Team, 2016).

Shape and space are explored in several parallel and connected strands: Properties of geometric figures and the logical connections between them, geometric measurement, and coordinate geometry.

Coordinate geometry is first introduced in fifth grade, and is an important way that geometry can be connected to algebra in ways that make clear the usefulness of algebraic tools and that illuminate meaning in many algebraic representations. In MIC 1 and 2, students use coordinates to prove simple geometric theorems, motivated by noticing features that seem to be true, and then trying to answer “Will that always be true? How can we know for sure?” In MIC 2, they switch between geometric and algebraic (equation) descriptions of conic sections when such different points of view are helpful to answering authentic questions about a context.

Geometric measurement is a strand that extends across all of kindergarten through grade twelve. In MIC 2, students use dissection and transformation arguments to informally justify formulas for circumference and area of circles and volume formulas for various three-dimensional figures. They explore the effect of scaling all linear measurements on area and volume measurements. All of these can be developed and used in the context of investigations that generate authentic questions for students: I wonder how much…?; I wonder how long…? etc. In MIC 3, geometric models of physical objects help to build models for data-driven or model-driven investigations.

While exploration of shape and space should be one of the easiest areas to motivate through investigations generating authentic questions, many students do not experience high school geometry this way. The strand that is the exploration of properties of geometric figures and the logical connections between them is the biggest culprit. One challenge is that *proving things that students consider obvious is not motivating or authentic*. As in most areas, much of the work of instructional designers (whether designing instructional materials or creating lesson plans) is to design real-world and mathematical activities in which students experience questions as authentic: that is, something they actually wonder about. After all, the mathematics of proof was originally developed to answer questions about which people were actually curious, and “it is useful for individuals to experience intellectual perturbations that are similar to those that resulted in the discovery of new knowledge” (Fuller, Rabin, and Harel, 2011). Thus, the mathematical activity of exploration of a context and deciding what might be true (by noticing patterns from examples) needs to be far more heavily represented in geometry class than is typical. Discussions of proof, causation, and correlation can happen across different classes, as students construct viable arguments and critique the reasoning of others.

Middle-school notions of congruence and similarity for plane figures are informal, based on work with transparencies or other tools that enable direct comparison. Experimentation with transformations continues in MIC 1, while definitions are made more precise. Congruence is defined in terms of rigid motions of the plane, and—because precisely finding and using rigid motions can be tedious—students show that triangles can be shown to be congruent using measurement instead. Triangle congruence criteria, demonstrated in terms of the rigid motion definition of congruence, need to answer an authentic question, perhaps as simple as “what’s the least information you can give your partner about your triangle, so that they can create a triangle that you are both certain is congruent to your original?” Similarly for geometric constructions, they must answer a question—“I wonder if…?” or “I wonder how….?”

MIC 2 introduces similarity by adding dilations to the rigid transformations that define congruence. Students prove a variety of geometric theorems, with a focus on understanding reasoning and not on a rigid form of proof. As mentioned in Content Connection 2, the relationship between lengths of corresponding sides of similar right triangles gives rise to the fact that their ratios are constant, and thus to names for those ratios (trigonometric functions).

MIC 3 includes investigations that make use of and reinforce geometric understanding developed in MIC 1 and 2. For instance, design challenges might have design constraints that call on plane geometry results, and many real-world modeling or data-driven investigations will involve physical objects that will have to be modeled mathematically to understand the system.

Content Connection 4: *Discovering Shape and Space* includes primarily the content of the CA CCSSM Conceptual Category *Geometry*. Investigations in Content Connection 4 will often involve quantities that change in related ways (e.g., lengths of sides in similar triangles) and will often require consideration of relationships between parts and wholes (e.g., the effect of scaling linear dimensions on area and volume measurements); thus, many investigations will pair Content Connection 4 with Content Connection 2 or Content Connection 3.

CC4: Discovering Shape and Space

MIC 1 Key Ideas	MIC 2 Key Ideas	MIC 3 Key Ideas
• Coordinates to prove simple theorems • Formalize transformation as function from plane to itself • Congruence in terms of rigid motions • Triangle congruence criteria from rigid motions definition	• Relationships between geometric and algebraic representations of conic sections • Dissection and transformations to justify area and volume formulas • Scaling's effect on area and volume • Similarity in terms of rigid motions plus dilations • Ratios of corresponding sides for similar right triangles (trigonometric functions)	• Geometry as necessitated by context to build and validate models

Key Mathematical Ideas to Promote Student Success In Introductory University Courses in Quantitative Fields[1]

Introduction

One of the important goals of K–12 mathematics is to prepare students for success in quantitative majors in college, should they choose to follow such a path. The route to equity in college-level education lies in good high school preparation. For a high school math pathway to provide this for a major, it needs to include the cumulative math knowledge and mathematical ways of thinking that are assumed in introductory courses for that major. Many foundational courses in quantitative majors require either calculus

[1] From comments submitted by Patrick Callahan (Callahan Consulting), Brian Conrad (Professor, Department of Mathematics, Stanford University), and Rafe Mazzeo (Professor, Department of Mathematics, Stanford University).

or topics and precise rigorous ways of thinking that are currently often learned on the path to calculus (e.g., facility with functions and algebra that arise in statistics). Moreover, students whose majors require calculus need to be prepared to learn it in college if they have not done so in high school. Educational developments of recent decades offer new ways to effectively deliver math curricula; these include group work, active-learning, and certain classroom technology. Motivation inspired by applications has always been a component of good mathematics teaching, and nowadays engaging contexts for many high school math topics can be drawn from business, computer science, data science, social sciences and even computer gaming design, complementing traditional motivation from the natural sciences and finance.

The list below focuses on topics prior to calculus, and the order of the items in the list has no significance. The final item (complex numbers) is asterisked because it is of tremendous importance in some quantitative fields (chemistry, engineering, physics, math) but not others (e.g., biology and economics).

1. **Representations of functions**. Functions as input-output laws can be expressed in many ways: algebraically as a formula, visually as a graph, a table of values, a recursive formula (such as for the factorial function), and so on. Exposure to the many ways of describing a function makes the concept more tangible (e.g., relating the graphs of $f(3x)$, $f(x+4)$, and $-2f(x)$ to that of $f(x)$), and helps students to grasp the broad importance and ubiquity of functions throughout mathematics and its applications. Computer programming uses functions everywhere, as do science, finance, engineering, and statistics.

2. **Familiarity with a variety of functions and manipulations with them**. This includes linear functions, the absolute value, polynomials, rational functions (relating back to comfort with fractions), exponential functions a^x, logarithmic functions $\log_b(x)$, and trigonometric functions (especially $\sin x$, $\cos x$, $\tan x$). The basic shape of their graphs should be known (e.g., a line for $2x-7$, periodic vertical asymptotes for $\tan x$, and how the graphs of x^2 and x^3 and 2^x differ), as should a feeling for their orders of magnitude (linear versus x^7 versus 2^x or $\log(x)$) and special algebraic rules for

exponentials and logarithms. This provides further opportunities to reinforce algebra material.

3. **Modeling with functions**. A particular mathematical model can be used and re-used to answer many quantitative questions. This reusability property is why mathematical models are worth formulating. Translating words into equivalent mathematical expressions and equations, and the reverse process, is fundamental to all uses of math to solve problems in the real world. This provides validation of a student's grasp of the meaning of mathematical concepts. It is also a different way of thinking than other more self-contained mathematical concepts. That translation process needs to be developed over time, and is not fully mastered before arrival in college but should be practiced in progressive levels of complexity starting from very early in a student’s education.

When expressing the information from a word problem in terms of mathematics, an essential step is often to introduce an appropriate function and to clarify hypotheses and definitions. Examples include exponentials for understanding pandemics and investment (geometric growth), logarithms for visualizing data that span many orders of magnitude, and sines and cosines to model periodic phenomena (giving contexts far beyond triangles for the relevance of such functions; the addition laws for sine and cosine encode phase-shifting). Data science, natural sciences, and computer science provide numerous examples of modeling with many types of functions. Even if a student won’t use a specific class of functions later on, exposure to a wealth of function types and their utility in high school makes the overall concept more grounded in reality. Students should also see how a mathematical model can be re-used to solve problems beyond the initial one that gave rise to the model (e.g., bankers and customers use the same compound interest model to answer different questions).

4. **Focus carefully enough on details to demonstrate good meta-cognition and to arrive at a reliable answer.** The ability to be self-critical and always check for consistency is important for the reliable application of mathematics and is only acquired through experience in solving mathematical problems. This includes finding one's mistake(s) when something has gone awry, and checking an answer “makes sense" in some basic ways (e.g., an area cannot be negative, and if a bank account is earning

interest then the value later should be larger). The use of technology does not eliminate the need for the latter, since erroneous information can be put into a computer: it is important to develop a sense of when an answer delivered by such means is way off base, signifying that the input was incorrect.

Mathematical problems can often be solved in a variety of ways, and it is both legitimate and important to often allow students the freedom to choose ways that make the most sense to them. However, an essential feature of the subject of mathematics is the concept of “correct answer" (in the sense of the outcome of a calculation, or solving equation(s) reliably) alongside attention to solution methods. The learning of mathematics should not overemphasize the answer to the exclusion of understanding of methodology, but the idea that many mathematical problems have a unique answer is important in many applications of mathematical models. Students should know that different exact solution methods *always* arrive at the same answer when no mistakes have been made and hypotheses remain unchanged, and that different approximate methods arrive at nearby answers. (Two collections of data in a mathematical model often differ, but measuring data is not solving an exact mathematical problem.)

The internal consistency of math is stronger than what is encountered in other areas of life, and is crucial for the reliability of engineering, the development and analysis of mathematical models, and the writing and trouble-shooting of computer programs. The modern technology and scientific progress everyone takes for granted (e.g., accurate GPS in planes and cars) relies crucially on mathematical problems having a “correct answer”, and students should appreciate the consistency of mathematics.

5. **Familiarity and comfort with symbolic manipulation, reinforced and extended in coursework after a first algebra course**. A reasonable level of comfort and confidence in the reading and manipulation of symbolic expressions is absolutely essential for reliable work with mathematics (even when using a computer to do number-crunching). This is not about grappling with very complicated expressions, but rather about reaching a level of comfort with symbolic expressions, applying basic manipulations with confidence, and knowing what one is doing with algebra and why one is doing it.

Examples include being able to work correctly with fractions (e.g., divide one fraction by another and reassemble as a single fraction), to read an algebraic expression (correctly interpreting order of operations), to plug in numbers for symbols to get numerical output, and to manipulate symbolic expressions in accordance with the laws of algebra: cancellation, factoring, working with square roots, exponents (e.g., express a ratio of powers of a common number as a single power of that number), etc. Students should also understand how to manipulate inequalities (such as in problems involving constraints or optimization).

Certain facts with whole numbers, such as $a(b+c) = ab + ac$ and $a^{n+m} = a^n a^m$, remain valid for broader types of numbers (negative, rational, and real). This wider validity should be highlighted, so students are aware of and become comfortable with its reliability. It is less important to know the name of such rules than it is to be aware of what rules are true; this is the mathematical counterpart of learning how to spell words. The laws of algebra should be seen as summarizing and abstracting facts from extensive concrete numerical experience, and not as arbitrary rules out of thin air to be memorized by rote. Indeed, students should learn that there is an inevitability to these rules, and that memorization is usually the least effective way to work with them. Developing this facility goes hand-in-hand with recognizing the falsity, in general, of statements such as "$(a+b)/(a+c) = b/c$" or passing sums through square roots or powers or absolute values. Plugging small numbers into a potential symbolic equality should be instinctive as a safety check (not as a justification, but as a way of sniffing out generally false statements).

6. **Working with and solving equations (and inequalities)**. This includes solving linear and quadratic equations in one variable, solving 2 linear equations in 2 unknowns, adding and subtracting equations from each other, and the visual meaning of such problems (crossing of 2 lines at a point, or finding where a graph crosses the horizontal coordinate axis). Solving exponential equations using logarithms is another important class of examples, as is knowing that often inequalities can be solved using analogues of methods for solving equations (along with some case-by-case work).

The key principle is to avoid a zoological chart of types of equations, and provide students with the means to gain experience using manipulation of both sides of an equation to isolate an unknown quantity to solve for it, and then (when relevant) to interpret the answer. It is important to be aware that sometimes an equation has no solutions or multiple solutions, and what that means in terms of a mathematical model. It is likewise important to recognize that an equation may be expressed in many equivalent forms (by applying a reversible operation to both sides).

7. **The mathematics of measurement**. This includes algebraic work with units of measurement (e.g., conversion among different units, using kilometers per hour, and recognizing that it makes no sense to add quantities with different units of measurement), and the development of an instinct to always use dimensional analysis (e.g., one cannot add a quantity measured in inches to one measured in square inches). Other crucial skills include ratios and percentages (as useful alternative language for certain types of work with fractions), and using scientific notation (reading it, and multiplying and dividing numbers written in this way).

These topics arise throughout applications of mathematics, and are an essential feature of answers to real-world word problems and questions in mathematical models; e.g., distances are never raw numbers (always some amount of kilometers, miles, etc.). Attention to units of measurement is necessary for meaningful answers to quantitative questions about the world.

8. **Trigonometry**. This admits different layers of understanding: the visual interpretation with right triangles (relating angles to lengths of sides based on similarity, including some special angles), the Law of Cosines for work with more general triangles, and the unit circle (explaining why sine and cosine relate to periodic phenomena, and visualizing that $\sin^2 x + \cos^2 x = 1$).

The traditional blizzard of “trigonometric identities" is not truly important (though it gives opportunities for experience with proofs in an algebraic setting). In data science a measure of “closeness" of vectors is a reinterpretation of the Law of Cosines, as is the notion of correlation between two data sets, and anyone who will do college-level work in engineering, physical sciences, or math (e.g., calculus) needs exposure to trigonometry up through the unit circle. For instance, some students desire to pursue computer graphics, such as for video game design, and this cannot be done without a solid command of trigonometry.

9. **Logical reasoning and justification**. Students should use careful arguments from hypotheses and definitions (and prior results) to arrive step-by-step at reliable conclusions, and get experience critiquing the reasoning of others. Although traditionally done in the context of plane geometry, such justification can also be done with algebra (e.g., mathematical induction to establish some formulas). Some students are predisposed towards visualization and others towards formulas, so exposure to the idea of proof or justification via reasoning in each of algebra and geometry makes principles more accessible.

What matters is practice with justifying steps in an argument, seeing logical reasoning used to arrive at results that are sometimes not evident (e.g., Pythagorean Theorem, some facts about angles inscribed in circles, and the formula for $1 + 2 + 3 + ... + n$), identifying flaws in an incorrect argument (e.g., overlooking division by 0 that masks a counterexample, making an algebra error, etc.), and knowing the internal consistency of correct mathematical results (i.e., two correct facts in math are *never* incompatible). Diagnosing bugs in computer programs requires the capacity for clear thinking that is provided only by experience with this aspect of mathematics. Proofs and justifications should be seen not as a mechanical cookbook of rules to follow, but as a reliable means of arriving at correct conclusions and gaining understanding. To the extent possible, students should see some logical arguments where the conclusion is an unexpected or surprising result.

10. **Geometry in the plane both visually and algebraically**. This includes a variety of facts about polygons, angles, lines, circles, relations between similar triangles, the Pythagorean Theorem, and equations expressing circles and lines in algebraic form via coordinate geometry.

Geometric knowledge with similar triangles is further enhanced later on by using appropriate trigonometric functions to compute side lengths of a right triangle when given an angle (important for computing distances) and relating arc length along a circle to the angle of a sector. Vectors in the plane and transformations of a plane (rotation, dilation, shearing, etc.) provide a connection between algebra and geometry (with parallelograms and triangles) that is of great importance in data science (e.g., linear algebra) and physics (and in work with complex numbers).

11. **Basic ideas from probability and statistics**. This includes independence of events, conditional probability, mean, variance, and learning from data. There is a vast array of applications of these ideas, illustrated by: coin tosses, heredity, the difficulty of testing for rare diseases, the prosecutor's fallacy, and finding the best-fit line for planar data (and related concepts: correlation coefficient, slope, y-intercept, etc.). Both log-log plots and specific probability distributions (such as the normal distribution, binomial distribution, and Poisson distribution) with their precise symbolic definitions extend and reinforce experience with exponentials, logarithms, and other concepts from algebra (as well as the notion of function).

12*. **Complex numbers**. This includes knowing how to add, subtract, multiply, and divide complex numbers (writing numbers in the standard form $a+bi$), and using complex numbers to solve a quadratic equation with real coefficients. The visual meaning of complex numbers is important, to provide a concrete interpretation of them. When trigonometry has been learned, the polar form $r\,(\cos(\vartheta) + i\,\sin(\vartheta))$ provides a valuable visual meaning for multiplication and reinforcement of some facts from trigonometry (e.g., addition laws for sine and cosine).

The topic of complex numbers extends experience with the universality of the laws of algebra and enhances mathematical maturity for appreciating the role of definitions in mathematics and learning broader conceptions of “algebra” later on (e.g., linear algebra in college, which pervades all quantitative modeling). It is fundamental for college-level work involving physics, chemistry, and engineering, and is closely related to some topics in computer science (e.g., Google’s PageRank algorithm, the algebraic systems used in error-correcting codes and encryption, and the discrete Fourier transform in machine learning and data analysis).

Links to Long Descriptions for Appendix A

Figure A.2: High School Algebra I 1 Big Ideas

The graphic illustrates the connections and relationships of some high school algebra mathematics concepts. Direct connections include:

- Model with Functions directly connects to: Features of Functions, Growth & Decay, Investigate Data, Systems of Equations, Function Investigations
- Features of Functions directly connects to: Growth & Decay, Systems of Equations, Function Investigations, Model with Functions
- Growth & Decay directly connects to: Features of Functions, Model with Functions, Function Investigations, Systems of Equations
- Systems of Equations directly connects to: Growth & Decay, Features of Functions, Model with Functions, Function Investigations
- Function Investigations directly connects to: Model with Functions, Features of Functions, Growth & Decay, Investigate Data, Systems of Equations
- Investigate Data directly connects to: Model with Functions, Function Investigations. Return to graphic.

Figure A.4: Big Ideas Map for Geometry

The graphic illustrates the connections and relationships of some high school geometry mathematics concepts. Direct connections include:

- Probability Modeling directly connects to: Fairness in Data
- Fairness in Data directly connects to: Probability Modeling
- Trig Explorations directly connects to: Triangle Congruence, Geometric Models, Triangle Problems, Geospatial Data, Circle Relationships, Points & Shapes

- Triangle Congruence directly connects to: Geometric Models, Triangle Problems, Transformations, Geospatial Data, Circle Relationships, Points & Shapes, Trig Explorations
- Geometric Models directly connects to: Triangle Problems, Transformations, Circle Relationships, Points & Shapes, Trig Explorations, Triangle Congruence
- Triangle Problems directly connects to: Geometric Models, Triangle Congruence, Transformations, Geospatial Data, Circle Relationships, Points & Shapes, Trig Explorations
- Transformations directly connects to: Geometric Models, Triangle Problems, Triangle Congruence, Geospatial Data, Circle Relationships, Points & Shapes
- Circle Relationships directly connects to: Geometric Models, Triangle Problems, Transformations, Geospatial Data, Triangle Congruence, Points & Shapes, Trig Explorations
- Points & Shapes directly connects to: Geometric Models, Triangle Problems, Transformations, Geospatial Data, Circle Relationships, Triangle Congruence, Trig Explorations
- Geospatial Data: Triangle Problems, Transformations, Triangle Congruence, Circle Relationships, Points & Shapes, Trig Explorations. Return to graphic.

Geometry

An illustration of the reasoning that corresponding parts being congruent implies triangle congruence, in which point A is translated to D, the resulting image of ΔABC is rotated so as to place B onto E, and finally, the image is then reflected along line segment DE to match point C to F. Return to graphic.

Figure A.7: High School Integrated 1 Big Ideas

The graphic illustrates the connections and relationships of some high school integrated mathematics concepts. Direct connections include:

- Systems of Equations directly connects to: Variability, Comparing Models, Modeling with Functions
- Correlation & Causation directly connects to: Variability, Comparing Models
- Variability directly connects to: Correlation & Causation, Comparing Models, Systems of Equations, Modeling with Functions, Building with Triangles
- Building with Triangles directly connects to: Variability, Comparing Models, Transformations & Congruence, Shapes in Structures, Modeling with Functions
- Composing Functions directly connects to: Transformations & Congruence, Shapes in Structures
- Modeling with Functions directly connects to: Building with Triangles, Variability, Comparing Models, Systems of Equations
- Shapes in Structures directly connects to: Transformations & Congruence, Building with Triangles, Composing Functions
- Transformations & Congruence directly connects to: Building with Triangles, Composing Functions, Shapes in Structures
- Comparing Models directly connects to: Correlation & Causation, Variability, Building with Triangles, Modeling with Functions, Systems of Equations.

  Return to graphic.

Figure A.9: Big Ideas Map for Integrated 2

The graphic illustrates the connections and relationships of some high school integrated mathematics concepts. Direct connections include:

- Function Representations directly connects to: Equations to Predict & Model, Polynomial Identities, Circle Relationships, Functions in the World, Trig Functions, Experimental Models & Functions
- Equations to Predict & Model directly connects to: Polynomial Identities, Circle Relationships, Trig Functions, Functions in the World, Transformations & Similarity, Experimental Models & Functions, Function Representations
- Polynomial Identities directly connects to: Geospatial Data, Circle Relationships, Trig Functions, Transformations & Similarity, Functions in the World,

Experimental Models & Functions, Function Representations, Equations to Predict & Model

- Geospatial Data directly connects to: Polynomial Identities, Functions in the World, Transformations & Similarity, Trig Functions, Circle Relationships
- Circle Relationships directly connects to: Geospatial Data, Polynomial Identities, Trig Functions, Transformations & Similarity, Functions in the World, Experimental Models & Functions, Function Representations, Equations to Predict & Model
- Trig Functions directly connects to: Geospatial Data, Circle Relationships, Polynomial Identities, Transformations & Similarity, Experimental Models & Functions, Function Representations, Equations to Predict & Model
- Transformations & Similarities directly connects to: Geospatial Data, Circle Relationships, Trig Functions, Polynomial Identities, Experimental Models & Functions, Equations to Predict & Model
- Experimental Models & Functions directly connects to: Circle Relationships, Trig Functions, Transformations & Similarity, Polynomial Identities, Function. Return to graphic.

Figure A.15: High School MIC 1 Big Ideas

The graphic illustrates the connections and relationships of some high school integrated mathematics concepts. Direct connections include:

- Systems of Equations directly connects to: Variability, Comparing Models, Modeling with Functions
- Correlation & Causation directly connects to: Variability, Comparing Models
- Variability directly connects to: Correlation & Causation, Comparing Models, Systems of Equations, Modeling with Functions, Building with Triangles

- Building with Triangles directly connects to: Variability, Comparing Models, Transformations & Congruence, Shapes in Structures, Modeling with Functions
- Composing Functions directly connects to: Transformations & Congruence, Shapes in Structures
- Modeling with Functions directly connects to: Building with Triangles, Variability, Comparing Models, Systems of Equations
- Shapes in Structures directly connects to: Transformations & Congruence, Building with Triangles, Composing Functions
- Transformations & Congruence directly connects to: Building with Triangles, Composing Functions, Shapes in Structures
- Comparing Models directly connects to: Correlation & Causation, Variability, Building with Triangles, Modeling with Functions, Systems of Equations
Return to graphic.

Figure A.17: Big Ideas Map for MIC 2

The graphic illustrates the connections and relationships of some high school integrated mathematics concepts. Direct connections include:

- Function Representations directly connects to: Equations to Predict & Model, Polynomial Identities, Circle Relationships, Functions in the World, Trig Functions, Experimental Models & Functions
- Equations to Predict & Model directly connects to: Polynomial Identities, Circle Relationships, Trig Functions, Functions in the World, Transformations & Similarity, Experimental Models & Functions, Function Representations
- Polynomial Identities directly connects to: Geospatial Data, Circle Relationships, Trig Functions, Transformations & Similarity, Functions in the World, Experimental Models & Functions, Function Representations, Equations to Predict & Model

- Geospatial Data directly connects to: Polynomial Identities, Functions in the World, Transformations & Similarity, Trig Functions, Circle Relationships
- Circle Relationships directly connects to: Geospatial Data, Polynomial Identities, Trig Functions, Transformations & Similarity, Functions in the World, Experimental Models & Functions, Function Representations, Equations to Predict & Model
- Trig Functions directly connects to: Geospatial Data, Circle Relationships, Polynomial Identities, Transformations & Similarity, Experimental Models & Functions, Function Representations, Equations to Predict & Model
- Transformations & Similarities directly connects to: Geospatial Data, Circle Relationships, Trig Functions, Polynomial Identities, Experimental Models & Functions, Equations to Predict & Model
- Experimental Models & Functions directly connects to: Circle Relationships, Trig Functions, Transformations & Similarity, Polynomial Identities, Function Representations, Equations to Predict & Model, The Shape of Distributions, Probability Modeling
- Probability Modeling directly connects to: The Shape of Distributions, Experimental Models & Functions
- The Shape of Distributions directly connects to: Probability Modeling, Experimental Models & Functions
- Functions in the world directly connects to: Functions Representations, Equations to Predict & Model, Polynomial Identities, Geospatial Data, Circle Relationships. Return to graphic.