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Mathematics Framework
Appendix A: Key Mathematical Ideas to Promote
Student Success in Introductory University Courses
in Quantitative Fields

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13 **Key Mathematical Ideas to Promote Student Success in**
14 **Introductory University Courses in Quantitative Fields¹**

15 One of the important goals of K–12 mathematics is to prepare students for success in
16 quantitative majors in college, should they choose to follow such a path. The route to
17 equity in college-level education lies in good high school preparation. For a high school
18 math pathway to provide good preparation for a major, it needs to include the
19 cumulative math knowledge and mathematical ways of thinking that are assumed in
20 introductory courses for that major. Many foundational courses in quantitative majors
21 require either calculus or topics and precise, rigorous ways of thinking that are currently
22 often learned on the path to calculus (e.g., facility with functions and algebra that arise
23 in statistics). Moreover, students whose majors require calculus need to be prepared to
24 learn it in college if they have not done so in high school. Educational developments of
25 recent decades offer new ways to effectively deliver math curricula. These include
26 group work, active learning, and certain kinds of classroom technology. Motivation
27 inspired by applications has always been a component of good mathematics teaching.
28 Today, engaging contexts for many high school math topics can be drawn from
29 business, computer science, data science, social sciences and even computer gaming
30 design, complementing traditional applications from the natural sciences and finance.

31 To promote success in introductory university courses in quantitative fields, students
32 should be exposed to key mathematical ideas. The list below focuses on topics prior to
33 calculus; the order of the items has no significance. The final item (complex numbers)
34 has an asterisk because it is of tremendous importance in some quantitative fields

¹ From comments submitted by Patrick Callahan (Callahan Consulting), Brian Conrad (Professor, Department of Mathematics, Stanford University), and Rafe Mazzeo (Professor, Department of Mathematics, Stanford University).

35 (chemistry, engineering, physics, math) but not others (e.g., biology and economics).
36 Students should:

37 **1. Understand representations of functions.** Functions as input-output laws can be
38 expressed in many ways: algebraically as a formula or as a recursive formula (such as
39 for the factorial function); visually as a graph or a table of values; and so on. Exposure
40 to the many ways of describing a function makes the concept more tangible (e.g.,
41 relating the graphs of $f(3x)$, $f(x+4)$, and $-2f(x)$ to that of $f(x)$) and helps students to grasp
42 the broad importance and ubiquity of functions throughout mathematics and their
43 applications. Computer programming uses functions extensively, as do science,
44 finance, engineering, and statistics.

45 **2. Be familiar with a variety of functions and manipulations with them.** Included
46 here are linear functions, the absolute value, polynomials, rational functions (relating
47 back to comfort with fractions), exponential functions a^x , logarithmic functions $\log_b(x)$,
48 and trigonometric functions (especially $\sin x$, $\cos x$, $\tan x$). Students should know the
49 basic shape of the functions' graphs (e.g., a line for $2x-7$, periodic vertical asymptotes
50 for $\tan x$, and how the graphs of x^2 and x^3 and 2^x differ) and have a sense of their orders
51 of magnitude (linear versus x^7 versus 2^x or $\log(x)$) as well as of special algebraic rules
52 for exponentials and logarithms. These topics provide further opportunities to reinforce
53 algebra material.

54 **3. Be familiar with modeling with functions.** A particular mathematical model can be
55 used and re-used to answer many quantitative questions. This reusability property is
56 why mathematical models are worth formulating. Translating words into equivalent
57 mathematical expressions and equations, along with the reverse process, are
58 fundamental to all uses of math to solve problems in the real world. This translation
59 capability provides validation of a student's grasp of the meaning of mathematical
60 concepts. It also represents a different way of thinking from that required by other, more
61 self-contained mathematical concepts. Translation capability needs to be developed
62 over time. It is not fully mastered before arrival in college but should be practiced at
63 progressive levels of complexity, starting very early in a student's education.

64 When expressing the information from a word problem in terms of mathematics, an
65 often essential step is to introduce an appropriate function and to clarify hypotheses and
66 definitions. Examples include exponentials for understanding pandemics and
67 investment (geometric growth), logarithms for visualizing data that span many orders of
68 magnitude, and sines and cosines to model periodic phenomena (giving contexts far
69 beyond triangles for the relevance of such functions; the addition laws for sine and
70 cosine encode phase-shifting). Data science, natural sciences, and computer science
71 provide numerous examples of modeling with many types of functions. Even if a student
72 won't use a specific class of functions later on, exposure in high school to a wealth of
73 function types and their utility gives the overall concept a firm grounding in reality.
74 Students should also see how a mathematical model can be re-used to solve problems
75 beyond the initial one that gave rise to the model (e.g., bankers and customers use the
76 same compound interest model to answer different questions).

77 **4. Know the importance of focusing carefully enough on details to demonstrate**
78 **good meta-cognition and to arrive at a reliable answer.** Being self-critical and
79 always checking for consistency are important habits for the reliable application of
80 mathematics and are only acquired through experience in solving mathematical
81 problems. These habits include finding one's mistake(s) when something has gone awry
82 and checking that an answer "makes sense" in basic ways (e.g., an area cannot be
83 negative; if a bank account is earning interest, then its value later should be larger). The
84 use of technology does not eliminate the need for the latter, since erroneous information
85 can be put into a computer. It is important to develop a sense of when an answer
86 delivered by technological means is way off base, signifying that the input was incorrect.

87 Mathematical problems can often be solved in a variety of ways, and it is both legitimate
88 and important to often allow students the choice of solving problems in ways that make
89 the most sense to them. However, an essential feature of mathematics is the concept of
90 "correct answer" (in the sense of the outcome of a calculation or solving equation(s)
91 reliably) alongside attention to solution methods. The learning of mathematics should
92 not emphasize "answer finding" to the exclusion of understanding of methodology, but
93 the idea that many mathematical problems have a unique answer is important in many
94 applications of mathematical models. Students should know that different exact solution

95 methods *always* arrive at the same answer when no mistakes have been made and
96 hypotheses remain unchanged and that different approximate methods arrive at nearby
97 answers. (Two collections of data in a mathematical model often differ, but measuring
98 data is not solving an exact mathematical problem.)

99 The internal consistency of math is stronger than what is encountered in other areas of
100 life, and that consistency is crucial for the reliability of engineering, the development and
101 analysis of mathematical models, and the writing and trouble-shooting of computer
102 programs. Scientific advances and modern technologies now taken for granted (e.g.,
103 accurate GPS in planes and cars) rely crucially on mathematical problems having a
104 “correct answer,” and students should appreciate this consistency inherent in
105 mathematics.

106 **5. Be familiar and comfortable with symbolic manipulation, skills reinforced and**
107 **extended in coursework after a first algebra course.** A reasonable level of comfort
108 and confidence in the reading and manipulation of symbolic expressions is absolutely
109 essential for reliable work with mathematics (even when using a computer to do
110 number-crunching). This does not mean grappling with very complicated expressions,
111 but rather is about reaching a level of comfort with symbolic expressions, applying basic
112 manipulations with confidence, and knowing what one is doing with algebra and why
113 one is doing it.

114 Examples include being able to work correctly with fractions (e.g., dividing one fraction
115 by another and reassembling as a single fraction), to read an algebraic expression
116 (correctly interpreting the order of operations), to plug in numbers for symbols to get
117 numerical output, and to manipulate symbolic expressions in accordance with the laws
118 of algebra—that is, to be comfortable with simplifying, factoring, working with square
119 roots and exponents (e.g., express a ratio of powers of a common number as a single
120 power of that number), etc. Students should also understand how to manipulate
121 inequalities (such as in problems involving constraints or optimization).

122 Certain facts with whole numbers, such as $a(b+c) = ab + ac$ and $a^{n+m} = a^n a^m$, remain
123 valid for broader types of numbers (negative, rational, and real). This wider validity
124 should be highlighted, so that students are aware of and become comfortable with its

125 reliability. It is less important to know the names of such rules than it is to be aware of
126 what rules are true; this is the mathematical counterpart of learning how to spell words.
127 The laws of algebra should be seen as summarizing and abstracting facts from
128 extensive concrete numerical experience, and not as arbitrary rules out of thin air to be
129 memorized by rote. Indeed, students should learn that there is an inevitability to these
130 rules, and that memorization is usually the least effective way to work with them.
131 Developing this facility goes hand-in-hand with recognizing the falsity, in general, of
132 statements such as " $(a+b)/(a+c) = b/c$ " or passing sums through square roots or powers
133 or absolute values. Plugging small numbers into a potential symbolic equality should be
134 instinctive as a safety check (not as a justification, but as a way of sniffing out generally
135 false statements).

136 **6. Be able to work with and solve equations (and inequalities).** This includes
137 solving linear and quadratic equations in one variable, solving two linear equations in
138 two unknowns, adding and subtracting equations from each other, and understanding
139 the visual meaning of such problems (crossing of two lines at a point, or finding where a
140 graph crosses the horizontal coordinate axis). Solving exponential equations using
141 logarithms is another important class of examples, as is knowing that often inequalities
142 can be solved using analogues of methods for solving equations (along with some case-
143 by-case work).

144 The key principle is to avoid a zoological chart of types of equations, and provide
145 students with the means to gain experience using manipulation of both sides of an
146 equation to isolate an unknown quantity to solve for it, and then (when relevant) to
147 interpret the answer. It is important to be aware that sometimes an equation has no
148 solutions or multiple solutions, and to know what that means in terms of a mathematical
149 model. It is likewise important to recognize that an equation may be expressed in many
150 equivalent forms (by applying a reversible operation to both sides).

151 **7. Understand the mathematics of measurement.** This includes algebraic work with
152 units of measurement (e.g., conversion among different units, using kilometers per hour,
153 and recognizing that it makes no sense to add quantities with different units of
154 measurement) and the development of an instinct to always use dimensional analysis

155 (e.g., one cannot add a quantity measured in inches to one measured in square inches).
156 Other crucial relevant skills include using ratios and percentages (as useful alternative
157 language for certain types of work with fractions) and scientific notation (reading it and
158 multiplying and dividing numbers written in this way).

159 These topics arise throughout applications of mathematics, and are an essential feature
160 of answers to real-world word problems and questions in mathematical models (e.g.,
161 distances are never raw numbers; they are always some amount of kilometers, miles,
162 etc.). Attention to units of measurement is necessary for meaningful answers to
163 quantitative questions about the world.

164 **8. Study trigonometry.** Trigonometry admits different layers of understanding: the
165 visual interpretation with right triangles (relating angles to lengths of sides based on
166 similarity, including some special angles), the Law of Cosines for work with more
167 general triangles, and the unit circle (explaining why sine and cosine relate to periodic
168 phenomena and visualizing that $\sin^2x + \cos^2x = 1$).

169 The traditional blizzard of “trigonometric identities” is not truly important (though it gives
170 opportunities for experience with proofs in an algebraic setting). In data science a
171 measure of “closeness” of vectors is a reinterpretation of the Law of Cosines, as is the
172 notion of correlation between two data sets, and anyone who will do college-level work
173 in engineering, physical sciences, or math (e.g., calculus) needs exposure to
174 trigonometry up through the unit circle. For instance, some students desire to pursue
175 computer graphics, such as for video game design, and this cannot be done without a
176 solid command of trigonometry.

177 **9. Be able to apply logical reasoning and justification.** Students should use careful
178 arguments from hypotheses and definitions (and prior results) to arrive step-by-step at
179 reliable conclusions. They also need experience critiquing the reasoning of others.
180 Although traditionally done in the context of plane geometry, such justification can also
181 be done with algebra (e.g., mathematical induction to establish some formulas). Some
182 students are predisposed towards visualization and others towards formulas, so
183 exposure to the idea of proof or justification via reasoning in both algebra and geometry
184 makes principles more accessible.

185 What matters is practice with justifying steps in an argument, seeing logical reasoning
186 used to arrive at results that are sometimes not evident (e.g., the Pythagorean
187 Theorem, some facts about angles inscribed in circles, and the formula for $1 + 2 + 3 + \dots$
188 $+ n$), identifying flaws in an incorrect argument (e.g., overlooking division by 0 that
189 masks a counterexample, making an algebra error, etc.), and knowing the internal
190 consistency of correct mathematical results (i.e., two correct facts in math are *never*
191 incompatible). Diagnosing bugs in computer programs requires the capacity for clear
192 thinking that is provided only by experience with this aspect of mathematics. Proofs and
193 justifications should be seen not as a mechanical cookbook of rules to follow, but as a
194 reliable means of arriving at correct conclusions and gaining understanding. To the
195 extent possible, students should see some logical arguments where the conclusion is
196 an unexpected or surprising result.

197 **10. Understand geometry in the plane, both visually and algebraically.** This
198 includes understanding a variety of facts about polygons, angles, lines, circles, relations
199 between similar triangles, the Pythagorean Theorem, and equations expressing circles
200 and lines in algebraic form via coordinate geometry.

201 Geometric knowledge with similar triangles is further enhanced later on by using
202 appropriate trigonometric functions to compute side lengths of a right triangle when
203 given an angle (important for computing distances) and relating arc length along a circle
204 to the angle of a sector. Vectors in the plane and transformations of a plane (rotation,
205 dilation, shearing, etc.) provide a connection between algebra and geometry (with
206 parallelograms and triangles) that is of great importance in data science (e.g., linear
207 algebra) and physics (and in work with complex numbers).

208 **11. Be familiar with basic ideas from probability and statistics.** These ideas include
209 independence of events, conditional probability, mean, variance, and learning from
210 data. There is a vast array of applications of these ideas, illustrated by coin tosses,
211 heredity, the difficulty of testing for rare diseases, the prosecutor's fallacy, and finding
212 the best-fit line for planar data (and related concepts, such as correlation coefficient,
213 slope, y -intercept, etc.). Both log-log plots and specific probability distributions (such as
214 the normal distribution, binomial distribution, and Poisson distribution) with their precise

215 symbolic definitions extend and reinforce experience with exponentials, logarithms, and
216 other concepts from algebra (as well as the notion of function).

217 **12.* Understand complex numbers.** This includes knowing how to add, subtract,
218 multiply, and divide complex numbers (writing numbers in the standard form $a+bi$), and
219 using complex numbers to solve a quadratic equation with real coefficients. The visual
220 meaning of complex numbers is important for providing a concrete interpretation of
221 them. When a student has learned trigonometry, the polar form $r(\cos(\vartheta) + i\sin(\vartheta))$
222 provides a valuable visual meaning for multiplication and reinforcement of some facts
223 from trigonometry (e.g., addition laws for sine and cosine).

224 The topic of complex numbers extends experience with the universality of the laws of
225 algebra and enhances mathematical maturity for appreciating the role of definitions in
226 mathematics and learning broader conceptions of “algebra” later on (e.g., linear algebra
227 in college, which pervades all quantitative modeling). The topic of complex numbers is
228 fundamental for college-level work involving physics, chemistry, and engineering and is
229 closely related to some topics in computer science (e.g., Google’s PageRank algorithm,
230 the algebraic systems used in error-correcting codes and encryption, and the discrete
231 Fourier transform in machine learning and data analysis).

232 * This item has an asterisk because it is of tremendous importance in some quantitative
233 fields (chemistry, engineering, physics, math) but not others (e.g., biology and
234 economics).

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